Mathematical Enrichment

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10.30 – 11.30

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Combinatorics

ie. counting finite sets/collections, lists, sequences...

Simplest counting principle:

"Pigeonhole principle":

Given $\underset{(m>n)}{n+1}$ "pigeons" (math. objects) and n "pigeonholes" (label, box) then at least one pigeonhole contains 2 pigeons.

Examples (1) We have 13 people. Then some two share a month of birth.

(2) Given 20 ~~(distinct)~~ whole numbers (integers) then ~~a~~ is a pair whose difference is divisible by 19.

Solution pigeons: the numbers

pigeonholes: remainders on division by 19:

$$\underbrace{0, 1, 2, \ldots, 18}_{19 \text{ pigeonholes.}}$$

$\Rightarrow$ Some pair a, b have the same remainder and so $a-b$ is divisible by 19.

(3) Given m integers $a_1, a_2, \ldots, a_m$ show that some sum of the form $a_k + a_{k+1} + \cdots + a_\ell$ for some $(k \leq \ell)$ is divisible by m. $(a_1, a_2, \ldots \underbrace{a_k, \ldots, a_\ell}_{\text{add these}}, \ldots a_m)$

("partial sums")

Solution

m pigeons: $P_1 = a_1$, $P_2 = a_1 + a_2$, $P_3 = a_1 + a_2 + a_3, \ldots, P_m = a_1 + \cdots + a_m$

m pigeonholes: remainders on div by m

Either some P_i has remainder 0, and is divisible by m (and we've done)

Or we only use the $m-1$ pigeonholes $1, \ldots, m-1$ and hence by the P.P. some two ~~lie in the same~~ have the same remainder:

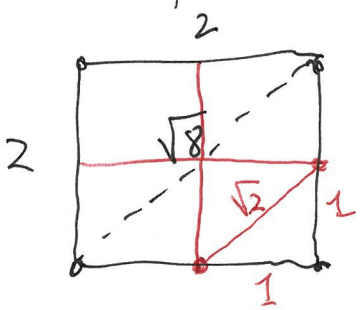
Say $P_k = a_1 + \cdots + a_k$

$k < \ell$ $P_\ell = a_1 + \cdots + a_k + \underline{a_{k+1} \cdots + a_\ell}$

$\Rightarrow$ m divides the difference

$P_\ell - P_k = a_{k+1} + \cdots + a_\ell$ as claimed.

(4) Given 5 points in a square of side 2, show some pair must lie within a distance $\sqrt{2}$



Solution pigeons: 5 points

pigeonholes: 4 squares of side 1.

By P.P, two points lie in same square of side 1.

(5) 101 integers chosen from $1, 2, \ldots, 200$

Show there is a pair one of which divides the other.

<u>Solution</u> Any integer $n = 2^s \cdot n_{odd}$ $(s \geq 0)$

pigeons: 101 numbers.

Pigeonholes: 100 odd numbers up to 200

$1, 3, \ldots, 199$

$n \downarrow n_{odd}$

By P.P., two m and n have same odd part, k say

$$m = 2^s k$$
$$n = 2^t \cdot k$$

WLOG $t \geq s \Rightarrow m \mid n$ $(n = 2^{t-s} \cdot m)$.

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(6) 101 distinct integers chosen from $1, 2, \ldots, 200$.

Show there is a pair with no common prime factor.

<u>Solution</u> pigeons: 101 numbers.

100 Pigeonholes: $\{1, 2\}, \{3, 4\}, \ldots, \{199, 200\}$

Classic Example (Erdös & Szekeres).

Given an ordered sequence $(n \geq 1)$

$$a_1, a_2, \ldots, a_{n^2+1}$$ of <u>real</u> numbers

then there is some subset of size $n+1$ which is either inc or dec (in the given order)

<u>Solution</u> For $k = 1, 2, \ldots, n^2+1$ let

$m_k :=$ length of largest increasing sequence beginning at a_k

If $m_k \geq n+1$ for some k, we are done.

[<u>Strong pigeonhole principle</u>:

m pigeons, n pigeonholes.

If $m > k \cdot n$ then some pigeonhole has $\geq k+1$ pigeons]

$m_1, \ldots, m_{n^2+1} \leq n$

$\Rightarrow$ $n+1$ must be equal. by strong P.P.

So $m_{k_1} = m_{k_2} = \cdots = m_{k_{n+1}}$ where $k_1 < k_2 < \cdots < k_{n+1}$

$\Rightarrow$ longest inc. sequence starting a_{k_1} = longest inc seq. at a_{k_2}.

$\Rightarrow$ $a_{k_1} > a_{k_2} > a_{k_3} > \cdots > a_{k_{n+1}}$ $\Leftarrow$ dec. seq of length $n+1$

Some Exercises

- Given 52 integers, show there is a pair whose sum or difference is divisible by 100.
  (What about if we ~~have~~ are given 51 integers?).

- Given any 17 integers show that it is possible to find 5 whose sum is divisible by 5.
- Generalize both problems!