UCD Maths Enrichment

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1. Let $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$ and let $S = \{4, 5, 9, 14, 23, 37\}$. Find two sets A and B with the properties
   - $A \cap B = \varnothing$
   - $A \cup B = C$
   - The sum of two distinct elements of A is not in S.
   - The sum of two distinct elements of B is not in S.

   (IrMO 2012, P1 Q1)

2. Find a positive integer n which has remainder 2 when divided by 7, remainder 2 when divided by 6, and remainder 6 when divided by 10.

3. n students sat a maths test with a certain number of problems. Each student solved exactly 5 problems, and each problem was solved by exactly 3 students. How many problems were on the test? For which n is this possible?

4. Find the smallest positive integer m such that $5m$ is an exact 5^{th} power, $6m$ is an exact 6^{th} power and $7m$ is an exact 7^{th} power.

   (IrMO 2013, P1 Q1)

5. In a tournament with N players, $N < 10$, each player plays once against each other player scoring 1 point for a win and 0 points for a loss. Draws do not occur. In a particular tournament only one player ended with an odd number of points and was ranked fourth. Determine whether or not this is possible. If so, how many wins did the player have?

   (IrMO 2011, P2 Q2)

6. Find, with proof, all polynomials f such that f has nonnegative integer coefficients, $f(1) = 8$ and $f(2) = 2012$.

   (IrMO 2012, P1 Q3)

7. Prove that there are two people in this room who have each shaken hands with the same number of people in this room.

8. Find, with proof, all pairs of integers (x, y) satisfying the equation

$$1 + 1996x + 1998y = xy$$

9. A rectangular array of positive integers has four rows. The sum of the entries in each column is 20. Within each row, all entries are distinct. What is the maximum possible number of columns?

(IrMO 2016, P2 Q2)

10. Let $x > 1$ be an integer. Prove that $x^5 + x + 1$ is divisible by at least two distinct prime numbers.

(IrMO 2012, P2 Q4)

11. Show that given 13 points in the plane with integer coordinates, there are three of them whose center of gravity has integer coordinates.

(Problem-Solving Methods in Combinatorics - Pablo Soberón, Problem 2.1)

12. (*) For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer} \\ a_n + 3 & \text{otherwise} \end{cases}$$

Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

(IMO 2017, Problem 1)

Hints

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1. Put 1 in A, and see what now can't be in A and must then be in B, and continue doing this for each element.
2. First find numbers m that satisfy
$$m \equiv 2 \pmod 6 \text{ and} \tag{1}$$
$$m \equiv 6 \pmod{10}. \tag{2}$$
3. Count the number of pairs (s, p) where s is a student and p is a problem. Count this quantity by student, and then by problem.
4. Prove that $5 \mid m$, $6 \mid m$ and $7 \mid m$. Write m as $m = 5^p \times 6^q \times 7^r \times A$ where $A \in \mathbb{Z}$.
5. There are $\binom{N}{2}$ games, and the same number of points. Which values of N make this total odd? Look at the remaining cases one by one.
6. Write the polynomial as a sum of unkown coefficients. Consider writing 2012 in binary.
7. For any given person, what are the possibilities for the number of people this person has shaken hands with?
8. $xy + ax + by + ab = (x + b)(y + a)$
Try adding something to both sides.
9. Count the sum of all the entries in the table, first by column, and by row. What this does say about n.
10. The key here is finding a factorisation of the term. Check that $x^2 + x + 1$ works.
11. Prove that for any 5 integers, there are three of them whose average is an integer. Try to reduce the given problem to this one. Interestingly, 13 is not the minimal integer for which the property in the original problem is true. Finding this is another interesting question.

12. Consider three cases depending on whether $a_0 \equiv 0$, 1 or 2 $\pmod 3$.
If $3 \mid a_0$ prove that $3 \mid a_n$ for all n. Consider what $0^2, 1^2$ and 2^2 are $\pmod 3$.
When $a_0 \equiv 2 \pmod 3$ consider if $m \in \mathbb{Z}$ can be a perfect square if $a_0 \equiv 2 \pmod 3$.
The remaining case is harder.

Solutions for IrMO problems can be found at –link–. http://www.irmo.ie/problems.html has a file called "Collected Problems" which has every IrMO paper from 1998 to 2014. The website Art of Problem Solving is a good resource for problems and solutions. The solution for IMO 2017 P1 should be there. If you have any questions regarding problems, solutions, or anything maths-related, feel free to contact us at cdohert2@tcd.ie or fortunma@tcd.ie.