UCD Mathematical Enrichment 10/03/18

Geometry: Similar Triangles

Warm-up problems

1. Let AP be tangent to the circle K at P and let B, C lie on K. Prove that
$$|\angle APB| = |\angle PCB|$$

2. Let P be a point not on the circle K and let the lines l_1, l_2 pass through P and cross K at A_1, B_1 and A_2, B_2 respectively. Using similar triangles, prove that
$$|PA_1||PB_1| = |PA_2||PB_2|$$

3. Let the circles K_1 and K_2 intersect at A and B. Let P be a point on AB and let l_1, l_2 be two lines through P. Suppose l_1 intersects K_1 at Q, R and l_2 intersects K_2 at S, T. Prove that $QRST$ is a cyclic quadrilateral. (Hint: Use the result from the problem 2.)

Intermediate problems

4. (IrMO 2013) The altitudes of a triangle $\triangle ABC$ are used to form the sides of a second triangle $\triangle A_1B_1C_1$. The altitudes of $\triangle A_1B_1C_1$ are then used to form the sides of a third triangle $\triangle A_2B_2C_2$. Prove that $\triangle A_2B_2C_2$ is similar to $\triangle ABC$.

5. (IrMO 2014) The square $ABCD$ is inscribed in a circle with centre O. Let E be the midpoint of AD. The line CE meets the circle again at F. The lines FB and AD meet at H. Prove $|HD| = 2|AH|$.

6. (BMO 2005, Round 1) Let $\triangle ABC$ be an acute-angled triangle, and let D, E be the feet of the perpendiculars from A, B to BC, CA respectively. Let P be the point where the line AD meets the semicircle constructed outwardly on BC, and Q be the point where the line BE meets the semicircle constructed outwardly on AC. Prove that $|CP| = |CQ|$.

Advanced problems

7. (IMO 2014) Points P and Q lie on side BC of acute-angled triangle $\triangle ABC$ so that $|\angle PAB| = |\angle BCA|$ and $|\angle CAQ| = |\angle ABC|$. Points M and N lie on lines AP and AQ respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the lines BM and CN intersect on the circumcircle of $\triangle ABC$.

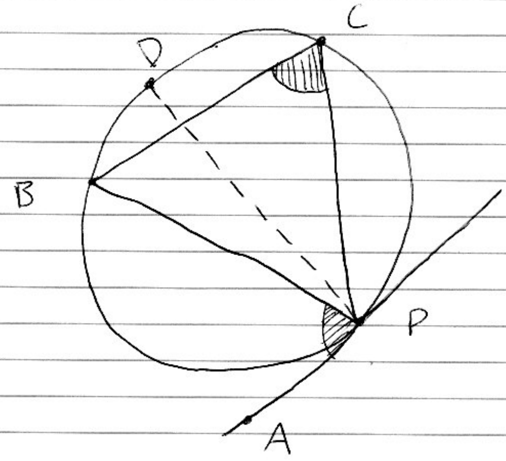
8. (IMO 2017) Let R and S be different points on a circle Ω such that RS is not a diameter. Let l be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle $\triangle JST$ intersects l at two distinct points. Let A be the common point of Γ and l that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Note on Solutions

Eventhough all of these solutions are simple, this certainly does not mean that all the problems are easy! Rather the solutions are in a condensed format that can only be achieved after many pages of rough work (for the more difficult problems). Keeping that in mind, you will not learn a lot from looking at the solutions without attempting the problems yourself. My recommendation to maximize learning is to do as much as you can by yourself and if you get stuck, look at the first one or two points of the solution and see if you can finish the problem from there.

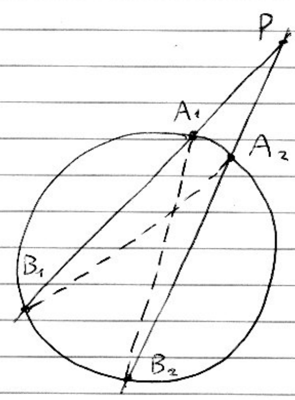
Good luck and enjoy the problems!

1.

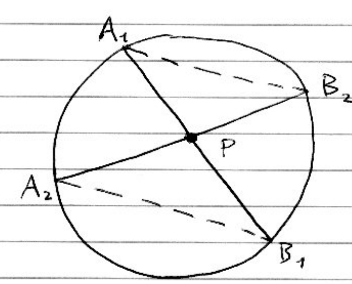


- Construct D so that DP is a diameter
- $|\angle DPA| = 90°$ (tangent).
- $|\angle DCP| = 90°$ (angle stands on diameter)
- $|\angle DPB| = |\angle DCB|$ (angles stand on same arc)
- $|\angle BCP| = |\angle DCP| - |\angle DCB| = 90° - |\angle DPB| = |\angle APB|$

2. (a)

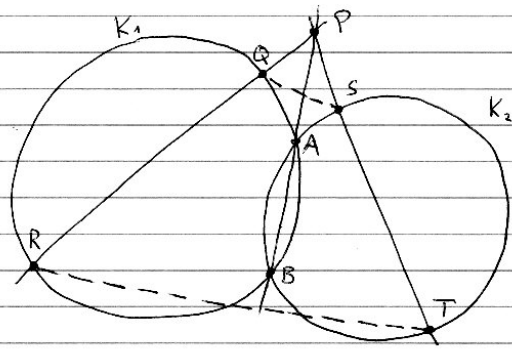


(b)



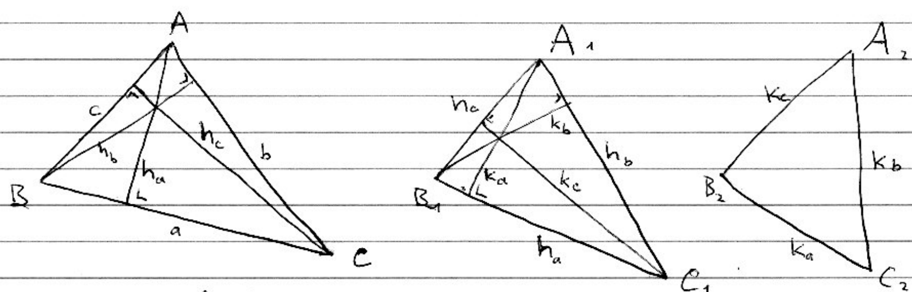
- Consider ΔA_1PB_2 and ΔA_2PB_1
- $|\angle A_1B_2P| = |\angle A_2B_1P|$ (stand on same arc)
- $|\angle A_1PB_2| = |\angle A_2PB_1|$ ((a) - same angle, (b) - symmetrically opposite angles)
- $\Delta A_1PB_2 \sim \Delta A_2PB_1$
- $\dfrac{|A_1P|}{|A_2P|} = \dfrac{|B_2P|}{|B_1P|} \Rightarrow |A_1P||B_1P| = |A_2P||B_2P|$

3.



- By 2., $|PQ||PR| = |PA||PB| = |PS||PT|$
- In the triangles ΔPQS and ΔPRT we have:
  - angles: $|\angle QPS| = |\angle RPT|$
  - sides: $\frac{|PQ|}{|PT|} = \frac{|PS|}{|PR|}$
- Therefore $\Delta PQS \sim \Delta PRT$
- $|\angle PQS| = |\angle PTR|$
- QRST is cyclic (exterior angle equals opposite interior angle)

4.



- area of triangle is given by:

$$\frac{1}{2} a h_a = \frac{1}{2} b h_b = \frac{1}{2} c h_c$$

- area of $\Delta A_1 B_1 C_1$ is similarly given by:

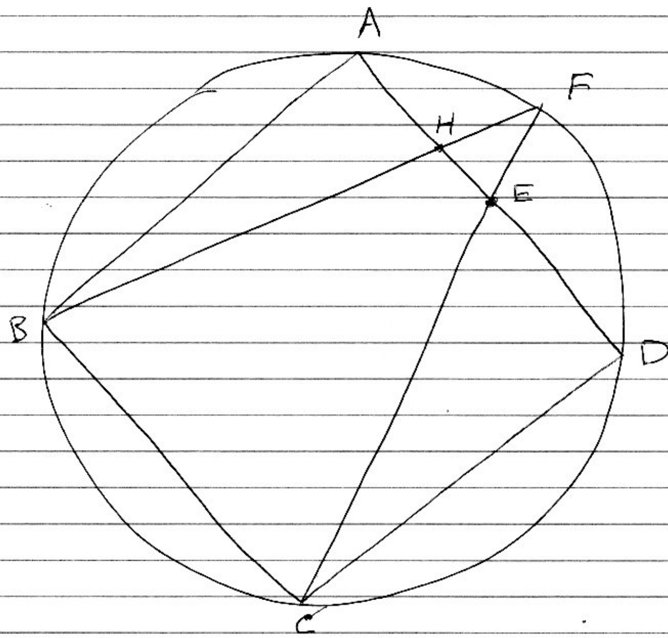
$$\frac{1}{2} h_a k_a = \frac{1}{2} h_b k_b = \frac{1}{2} h_c k_c$$

- divide first set of equations by second:

$$\frac{\frac{1}{2} a h_a}{\frac{1}{2} h_a k_a} = \frac{\frac{1}{2} b h_b}{\frac{1}{2} h_b k_b} = \frac{\frac{1}{2} c h_c}{\frac{1}{2} h_c k_c}$$

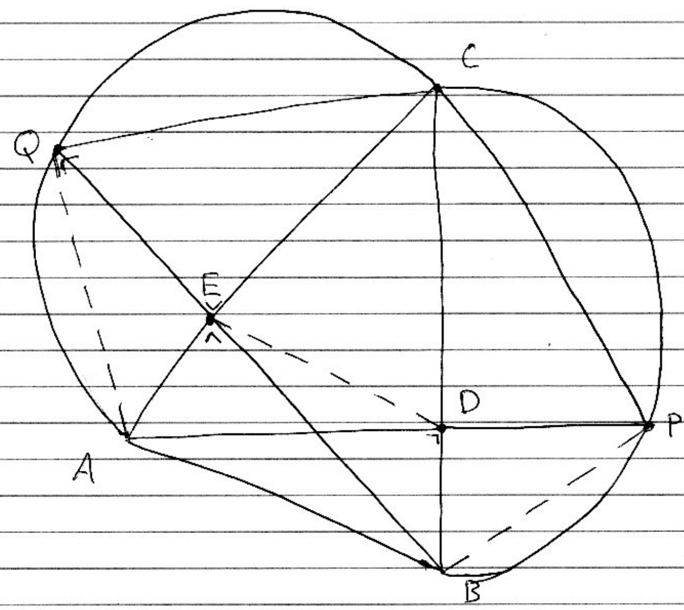
$$\Rightarrow \frac{a}{k_a} = \frac{b}{k_b} = \frac{c}{k_c}$$

5.



- Let the sidelength of the square be x.
- Notice that it is sufficient to find $|HE|$ in terms of x and we will be able to find $|AH|/|HD|$ from there
- $\triangle FHE \sim \triangle FBC$ $\quad (HE \parallel BC)$
- $\dfrac{|HE|}{|BC|} = \dfrac{|FH|}{|FB|} = \dfrac{|FE|}{|FC|}$
- The last ratio is easiest to calculate.
- $|FE||EC| = |AE||ED| = (\frac{1}{2}x)^2$
- $|EC|^2 = |CD|^2 + |DE|^2 = x^2 + (\frac{1}{2}x)^2 = \frac{5}{4}x^2$
- $\dfrac{|FE|}{|FC|} = \dfrac{|FE|}{|FE|+|EC|} = \dfrac{(\frac{1}{2}x)^2\frac{1}{|EC|}}{(\frac{1}{2}x)^2\frac{1}{|EC|} + |EC|} = \dfrac{(\frac{1}{2}x)^2}{(\frac{1}{2}x)^2 + |EC|^2} = \dfrac{1}{6}$
- $\dfrac{|AH|}{|HD|} = \dfrac{\frac{1}{2}x - |HE|}{\frac{1}{2}x + |HE|} = \dfrac{\frac{1}{2} - \frac{|HE|}{x}}{\frac{1}{2} + \frac{|HE|}{x}} = \dfrac{\frac{1}{2} - \frac{1}{6}}{\frac{1}{2} + \frac{1}{6}} = \dfrac{1}{2}$

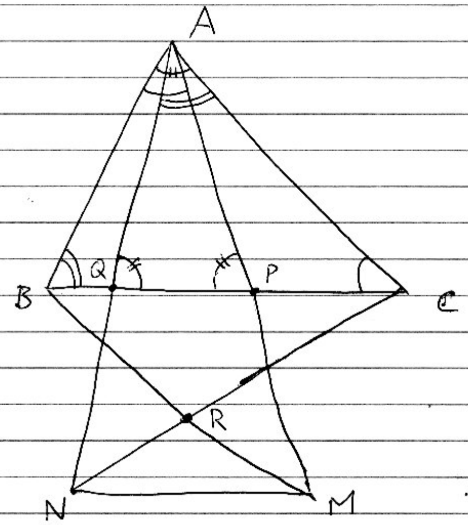
6.



- $|\angle AQC| = |\angle QEC| = 90^\circ$ (semicircle)
- ΔAQC and ΔQEC also share angle at C.
- Therefore $\Delta AQC \sim \Delta QEC$
- $\dfrac{|QC|}{|EC|} = \dfrac{|AC|}{|QC|} \Rightarrow |QC|^2 = |EC||AC|$
- Similarly, $|PC|^2 = |CD||CB|$
- ΔBEC and ΔADC both have a right angle and share angle at C
- Therefore $\Delta BEC \sim \Delta ADC$

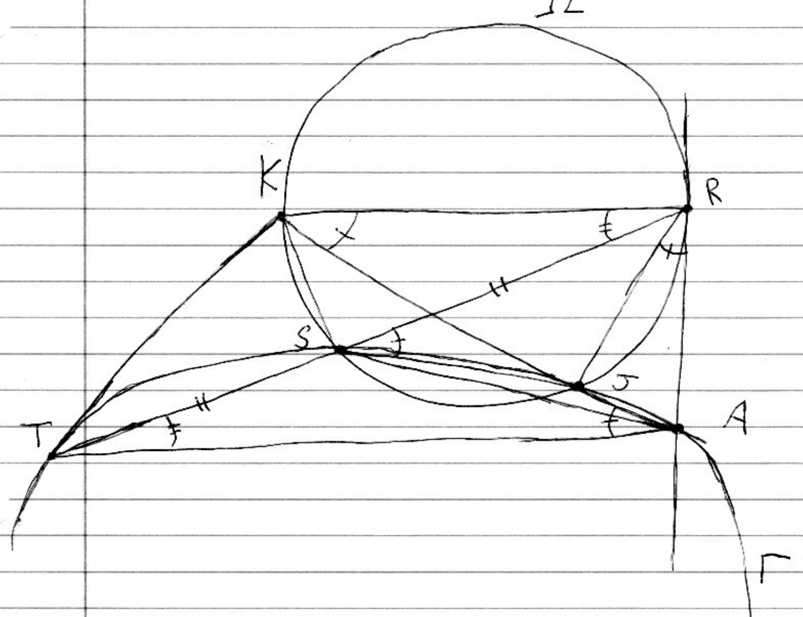
$\dfrac{|BC|}{|AC|} = \dfrac{|CE|}{|CD|} \Rightarrow |CD||CB| = |EC||AC| \Rightarrow |QC| = |PC|$

7.



- $|\angle AQC| = |\angle APB| = |\angle BAC|$
- $\triangle AQC \sim \triangle APB$
- $\dfrac{|AQ|}{|BP|} = \dfrac{|QC|}{|AP|} = \dfrac{|AC|}{|AB|}$
- $\Rightarrow \dfrac{|QN|}{|BP|} = \dfrac{|QC|}{|PM|}$
- Also $|\angle NQC| = |\angle BPM|$, so $\triangle QNC \sim \triangle BPM$
- $|\angle PBM| = |\angle QNC| = \alpha$, $|\angle QCN| = |\angle BMP| = \beta$
- $\alpha + \beta = |\angle QNC| + |\angle QCN| = |\angle QNC| + |\angle CNM|$ (since $BC \parallel NM$)
- $\alpha + \beta = |\angle QNM| = |\angle AQP| = |\angle BAC|$
- $|\angle BRC| = 180^\circ - |\angle MBP| - |\angle NCQ|$
  $= 180^\circ - \alpha - \beta$
  $= 180^\circ - |\angle BAC|$
- $ABCR$ is a cyclic quadrilateral

8.



- Since RA is tangent to Ω,
  $|\angle JRA| = |\angle JKR| = |\angle JSR|$ (Problem 1.)
- Since $AJST$ is cyclic,
  $|\angle JSR| = |\angle JAT|$
- Therefore $|\angle JAT| = |\angle JKR|$, so $KR \parallel TA$
- $|\angle ATR| = |\angle KRS|$ (alternate angles)
- Since RA is tangent to Ω,
  $|\angle SRA| = |\angle SKR|$
- Using the last two points, $\Delta KSR \sim \Delta ART$
- To show that KT is tangent to Γ, it is sufficient to show that
  $|\angle KTR| = |\angle SAT|$
- To prove that, it is sufficient to prove that
  $\Delta KTR \sim \Delta SAT$
- We already have one angle so we only need to show that the ratio of two pairs of sides is equal
- The easiest pair is:
  $\frac{|ST|}{|KR|}$ and $\frac{|TA|}{|TR|}$
- Note that $|ST| = |RS|$ (given)
- From $\Delta KSR \sim \Delta ART$, we get
  $\frac{|RS|}{|TA|} = \frac{|KR|}{|TR|}$

  and we are done

Resources

General Problem solving (Easy – Medium)

Brilliant.org

"Amusements in Mathematics", Henry Dudeney

Olympiad-style problems (Medium – Difficult)

Past papers (IrMO, BMO, IMO etc.)

"Problem-Solving Strategies", Arthur Engel

"Mathematical Olympiad Treasures", T. Andreescu, B. Enescu

"101 Problems in Algebra"
"102 Combinatorial Problems"
"103 Trigonometry Problems"
"104 Number Theory Problems"
} T. Andreescu, Z. Feng