MATHEMATICAL ENRICHMENT / OLYMPIAD TRAINING

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INEQUALITIES

BASIC INEQUALITY: $x^2 \geq 0$ and equality holds if and only if $x = 0$.

Usually x is a real number with $x \geq 0$.

Let's apply this to $(x-y)^2 \geq 0$ where $x \geq 0$, $y \geq 0$ are two real numbers

$$x^2 - 2xy + y^2 \geq 0$$

$$x^2 + y^2 \geq 2xy$$

equality holds if and only if $x = y$.

Example Show $x^2 + y^2 + z^2 \geq xy + yz + zx$ $\quad x \geq 0, y \geq 0, z \geq 0$

Solution:

$$(x-y)^2 \geq 0 \Rightarrow x^2 + y^2 \geq 2xy$$

$$(y-z)^2 \geq 0 \Rightarrow y^2 + z^2 \geq 2yz$$

$$(z-x)^2 \geq 0 \Rightarrow z^2 + x^2 \geq 2zx$$

Add $\quad 2x^2 + 2y^2 + 2z^2 \geq 2xy + 2yz + 2xz$

Divide by 2.

Equality holds if and only if $x = y = z$.

Example Show $\dfrac{a^2}{a+b} \geq \dfrac{3a-b}{4}$ where $a > 0$, $b > 0$

Solution:

$$(a-b)^2 \geq 0$$

$$\Rightarrow a^2 + b^2 \geq 2ab$$

$$\Rightarrow a^2 \geq 2ab - b^2$$

Add $3a^2$ $\Rightarrow$ $4a^2 \geq 3a^2 + 2ab - b^2 = (3a-b)(a+b)$

Divide by 4, and $a+b$, $\dfrac{a^2}{a+b} \geq \dfrac{3a-b}{4}$

Rough work

$$4a^2 \geq (3a-b)(a+b) = 3a^2 + 2ab - b^2$$

$$a^2 \geq 2ab - b^2$$

$$a^2 + b^2 \geq 2ab$$

Example ② Show $\dfrac{a^2}{a+b} + \dfrac{b^2}{b+c} + \dfrac{c^2}{c+a} \geq \dfrac{a+b+c}{2}$ $\quad a>0, \; b>0, \; c>0.$

Solution: By previous example,

$$\begin{aligned} \text{L.H.S.} &\geq \frac{3a-b}{4} + \frac{3b-c}{4} + \frac{3c-a}{4} \\ &= \frac{3a-b+3b-c+3c-a}{4} \\ &= \frac{2a+2b+2c}{4} \\ &= \frac{a+b+c}{2}. \end{aligned}$$

The arithmetic mean of x and y is $\dfrac{x+y}{2}$

The geometric mean of x and y is $\sqrt{xy}$

The AM – GM inequality says AM $\geq$ GM

$$\boxed{\frac{x+y}{2} \geq \sqrt{xy}}$$

here $x>0$, $y>0$.
equality holds iff $x=y$

Proof: $(\sqrt{x}-\sqrt{y})^2 \geq 0 \implies x - 2\sqrt{x}\sqrt{y} + y \geq 0$
$\implies x+y \geq 2\sqrt{xy}.$

Example if $x>0$ show $x + \frac{1}{x} \geq 2$ and equality holds iff $x=1$

Solution: Apply AM – GM with $y = \frac{1}{x}$

we get $\dfrac{x+\frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} = 1$

Multiply by 2 $\quad x + \frac{1}{x} \geq 2.$

Equality holds iff $x = \frac{1}{x}$

$x^2 = 1$

$\implies x = 1$ because $x>0.$

Exercise (3) Show $\frac{a}{b} + \frac{b}{a} \geq 2$ where $a > 0$
$b > 0$.

Example Show $(a+b)(b+c)(a+c) \geq 8abc$ where $a > 0$
$b > 0$
$c > 0$.

Solution : $\frac{a+b}{2} \geq \sqrt{ab}$ $\frac{b+c}{2} \geq \sqrt{bc}$ $\frac{a+c}{2} \geq \sqrt{ac}$

Multiply.

$$\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{a+c}{2}\right) \geq \sqrt{ab}\sqrt{bc}\sqrt{ac}$$

$$= \sqrt{a^2b^2c^2}$$

$$\frac{(a+b)(b+c)(a+c)}{8} = abc$$

Multiply by 8.

The AM-GM inequality for n numbers is

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

equality holds iff $a_1 = a_2 = \cdots = a_n$.

Example Show $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$ $x > 0$
$y > 0$
$z > 0$.

use AM-GM with $n = 3$.

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3\sqrt[3]{\frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}} = 3$$

Example Show $x^6 + y^6 + 4 \geq 6xy$ $x > 0$
$y > 0$.