Enumerative Combinatorics –

In other words, Counting!

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Permutations and Combinations

Problem 1.1 How many ways can the letters of the word MATHS be arranged in a row?

There are:

- 5 choices for the first letter;
- Then, 4 choices remaining for the second letter;
- Then, 3 choices remaining for the third letter;
- Then, 2 choices remaining for the fourth letter;
- Then, only 1 choice remaining for the fifth letter.

In total, the number of arrangements (or *permutations*), is

$$5! \ = \ 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

In general, the number of ways in which n different objects can be arranged in a row is

$$n! = n \times (n-1) \times (n-2) \cdots \times 3 \times 2 \times 1.$$

This notation is called "n factorial".

Problem 1.2 How many ways can the letters of the word HAPPY be arranged in a row?

Here the two letters P may be regarded as identical. If they were not identical, we would have 120 permutations as before. But each permutation is counted twice. Therefore the number of different permutations is

$$\frac{5!}{2!} = 60.$$

Problem 1.3 How many ways can the letters of the word HAPPPY be arranged in a row?

By a similar reasoning to the above, the answer is

$$\frac{6!}{3!} = 120.$$

If a set $A = \{a_1, a_2, \ldots, a_k\}$, then the size of A is denoted $|A|$ or $\#A$.

For example, if $A = \{2, 6, 7\}$, then $|A| = 3$.

Definition 1.4 For $n \geq 1$ and $0 \leq k \leq n$, the number of subsets of the set $\{1, 2, 3, \ldots, n\}$ of size k is

$$\binom{n}{k} := |\{A \subseteq \{1, 2, \ldots, n\} : A \text{ has } k \text{ elements}\}| .$$

Notice that choosing a subset is equivalent to choosing a team of k people from n people.

The numbers $\binom{n}{k}$ are called *binomial coefficients* and are central in enumerative combinatorics (counting). For completeness, we also define $\binom{0}{0} = 1$.

Example 1.5 For all natural numbers $n \geq 0$, $\binom{n}{0} = \binom{n}{n} = 1$.

It is easy to see that $\binom{n}{0} = 1$ for all numbers n. This is because there is only one subset of the set $\{1, 2, \ldots, n\}$ of size 0. It is the empty set $\emptyset$.

Similarly $\binom{n}{n} = 1$ for all natural numbers n because there is only one subset of $\{1, 2, \ldots, n\}$ of size n, namely the set $\{1, 2, \ldots, n\}$ itself.

Example 1.6 If we write down all of the 3-element subsets of $\{1, 2, 3, 4, 5\}$ we get

$$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\},$$
$$\{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}.$$

Thus we have $\binom{5}{3} = 10$.

How do we enumerate $\binom{n}{k}$ without constructing all the possible sets?

Let's go back to our example. Notice that choosing a subset is equivalent to choosing a team of 3 people from 5 people. We can arrange the 5 players in a row, and the three players who appear first in the row can be on the team. There are $5!$ arrangements for the row, but notice that every team will be counted $2! \times 3!$ times, since the three people on the left can be rearranged as we like (in $3!$ ways) without changing the team, and the last two people can also be rearranged as we like (in $2!$ ways) without changing the team.

We conclude that

$$\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10.$$

The same shows that in general:

$$\binom{n}{k} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k \times (k-1) \times \cdots \times 2 \times 1} = \frac{n!}{k!\,(n-k)!}. \tag{1}$$

Problem 1.7 In how many ways can we choose a team of 3 people from 8 people?

The answer is

$$\binom{8}{3} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = \frac{8!}{3!\,5!} = 56.$$

We can arrange the numbers $\binom{n}{k}$ to form Pascal's triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & \vdots & & & & & & & & \vdots &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & \vdots & & & & & & \vdots & &
\end{array}
$$

Some properties of the binomial coefficients:

- For all natural numbers n and k with $n \geq 0$ and $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Specifying which k people we choose for the team is exactly the same as specifying the $n-k$ people we leave behind (i.e., who are *not* on the team).

Problem 1.8 Find an alternative proof of this fact using Equation (1).

- For all natural numbers n, k where $0 \leq k < n$,

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Suppose we have $n+1$ people, including Tom. We want to choose a team of $k+1$ people out of these. There are $\binom{n+1}{k+1}$ ways to do this. Now, let's count these teams in a different way.

If we include Tom, then we have n people left over and we need to choose k. There are $\binom{n}{k}$ ways to do this.

If we do *not* choose Tom, then we have n people left over and we need to choose $k+1$. There are $\binom{n}{k+1}$ ways to do this.

We conclude that

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Problem 1.9 Find an alternative proof of this fact using Equation (1).

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Problem 1.10 How many ways can the letters of the word MISSISSIPPI be arranged in a row?

Problem 1.11 In a set there are n different blue objects and m different red objects. How many pairs of objects of the same colour can be made?

Problem 1.12 What is the value of

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} \; ?$$

$$\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} \; ?$$

The Binomial Theorem:

Theorem 1.13 For all natural numbers n and real numbers x,

$$\sum_{i=0}^{n} \binom{n}{i} x^i \;=\; (1+x)^n.$$

The L.H.S. is 'sigma notation' for the expression

$$\binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n.$$

Example 1.14 For $n = 2$ we have

$$\binom{2}{0} + \binom{2}{1} x + \binom{2}{2} x^2 \;=\; (1+x)^2.$$

This is easily verified by expanding the R.H.S.

Example 1.15 Substituting $n = 4$ and $x = 1$ above, we find that

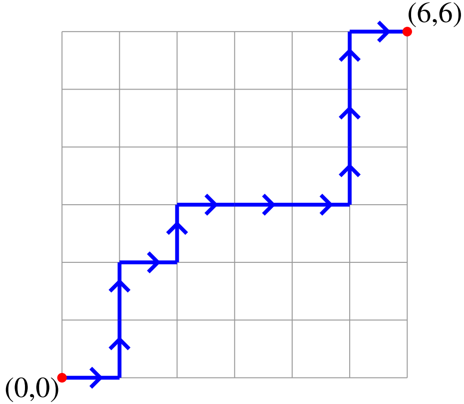
$$\begin{aligned}
&\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} \\
&= \binom{4}{0}1^0 + \binom{4}{1}1^1 + \binom{4}{2}1^2 + \binom{4}{3}1^3 + \binom{4}{4}1^4 \\
&= (1+1)^4 = 2^4 = 16.
\end{aligned}$$

Example 1.16 Substituting $n = 4$ and $x = -1$ above, we find that

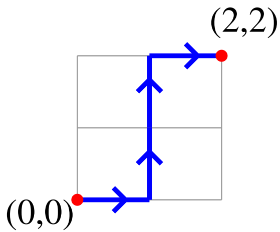
$$\begin{aligned}
&\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} \\
&= \binom{4}{0}(-1)^0 + \binom{4}{1}(-1)^1 + \binom{4}{2}(-1)^2 + \binom{4}{3}(-1)^3 + \binom{4}{4}(-1)^4 \\
&= (1+(-1))^4 = 0^4 = 0.
\end{aligned}$$

North-east Lattice Paths

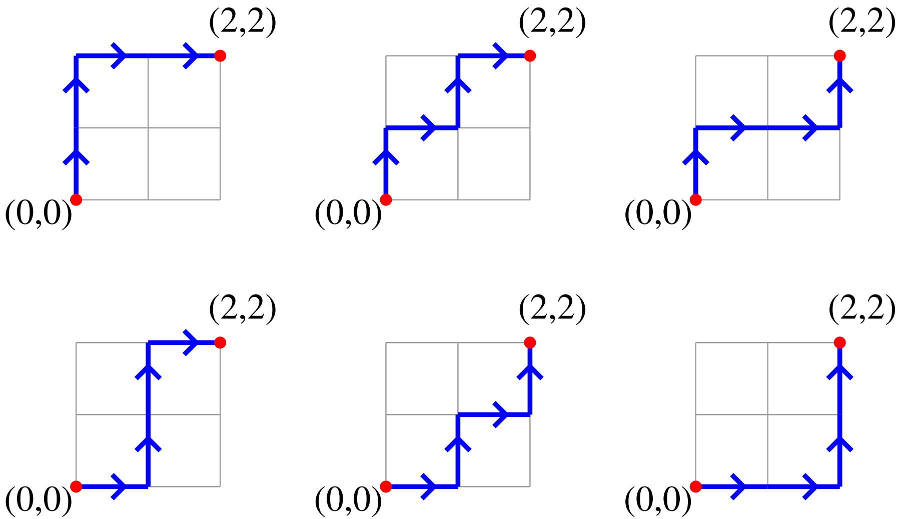
Problem 1.17 How many paths are there from the point $(0, 0)$ to the point $(6, 6)$ which are made up of East (1,0) and North (0,1) steps?



Lets consider the easier question: how many such paths are there from $(0, 0)$ to $(2, 2)$?



Well, we can go through them one-by-one:



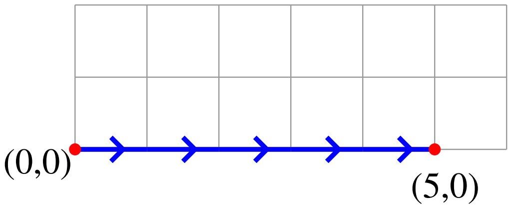
So the answer is there are 6 North-East paths from (0,0) to (2,2).
We counted these by the exhaustive method of producing all such paths. Is there a nicer way to do this so that we do not have to construct all the paths each time the numbers are changed?

In other words, can we find a formula for the number of such paths from $(0,0)$ to (n,n)?

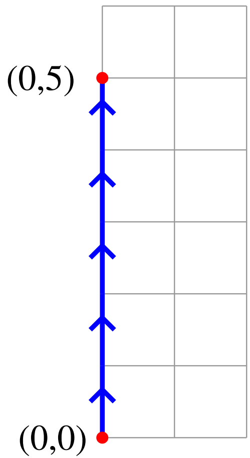
To attack such a problem, it is normally a good idea to give a 'name' to the quantity which we seek.

Let us write $f(a, b)$ for the number of North East paths from $(0, 0)$ to the point (a, b). Of course we assume a and b to be non-negative integers. Using this notation, we summarise the previous calculation by $f(2, 2) = 6$.

Question 1: How many North-East paths are there from $(0, 0)$ to $(a, 0)$? Well the only way to get to $(a, 0)$ from $(0, 0)$ is to take exactly a East steps. Therefore there is only one such path. So $f(a, 0) = 1$ for all integers a.



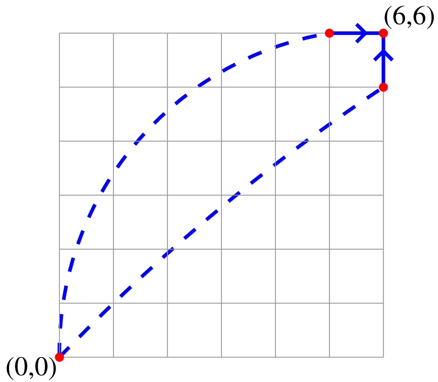
Question 2: How many North-East paths are there from $(0,0)$ to $(0,b)$? Well the only way to get to $(0,b)$ from $(0,0)$ is to take exactly b North steps. Therefore there is only one such path. So $f(0,b) = 1$ for all integers b.



Question 3: Suppose (a, b) is a point with $a, b > 0$. How do we end up at the point (a, b)? In other words, what happens just before we get to (a, b)?

Well since we take only North and East steps, either

- we are at $(a-1, b)$ and take a final East (1,0) step, or
- we are at $(a, b-1)$ and take a final North (0,1) step.



Every path from $(0,0)$ falls into one of the above categories. The number of paths from (0,0) to (a, b) is thus the number of paths from $(0,0)$ to $(a-1, b)$ plus the number of paths from $(0,0)$ to $(a, b-1)$. Therefore, for $a, b > 0$,

$$f(a, b) = f(a-1, b) + f(a, b-1).$$

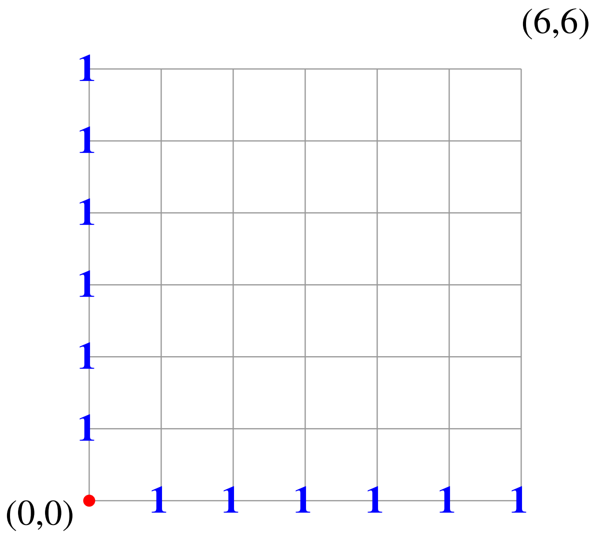
This is good news, we have a recursion for finding $f(a, b)$ without the need to construct paths any more:

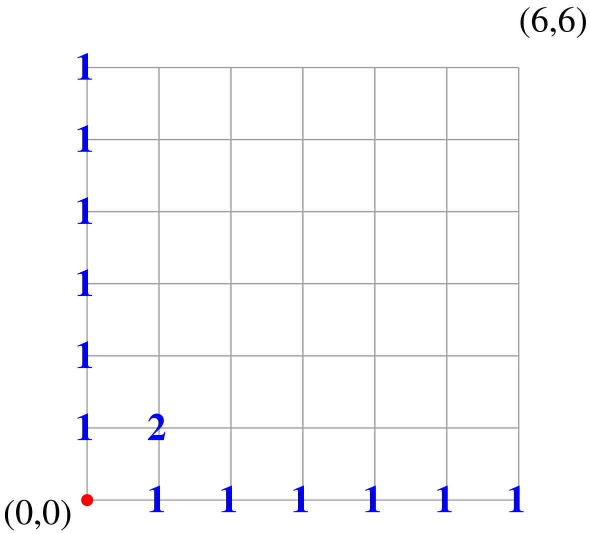
$$f(a,b) = \begin{cases} 1 & \text{if } a = 0 \\ 1 & \text{if } b = 0 \\ f(a-1,b) + f(a,b-1) & \text{if } a, b > 0. \end{cases}$$

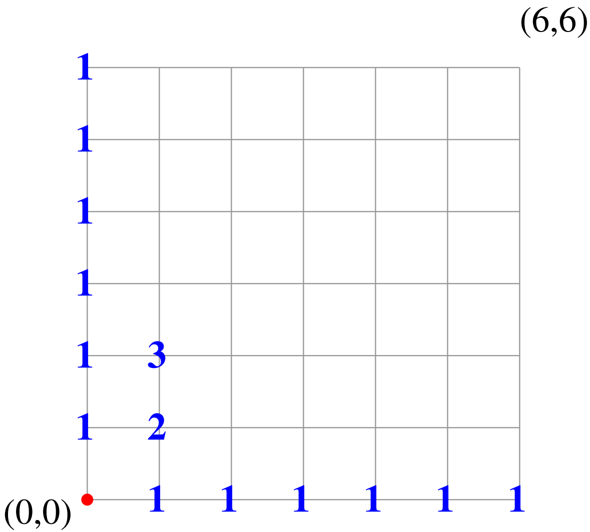
We may check this with the answer we got before:

$$\begin{aligned} f(2,2) &= f(1,2) + f(2,1) \\ &= (f(0,2) + f(1,1)) + (f(1,1) + f(2,0)) \\ &= (1 + f(1,1)) + (f(1,1) + 1) \\ &= 2f(1,1) + 2 \\ &= 2(f(0,1) + f(1,0)) + 2 \\ &= 2(1+1) + 2 \\ &= 6. \end{aligned}$$

A more visual way to make this calculation to do would be to insert at each entry (a, b) of the grid the number $f(a, b)$.


(6,6)
1
1
1
1
1
1
(0,0)
1 1 1 1 1 1


(6,6)
1
1
1
1
1
1 2
(0,0) 1 1 1 1 1 1


(6,6)
1
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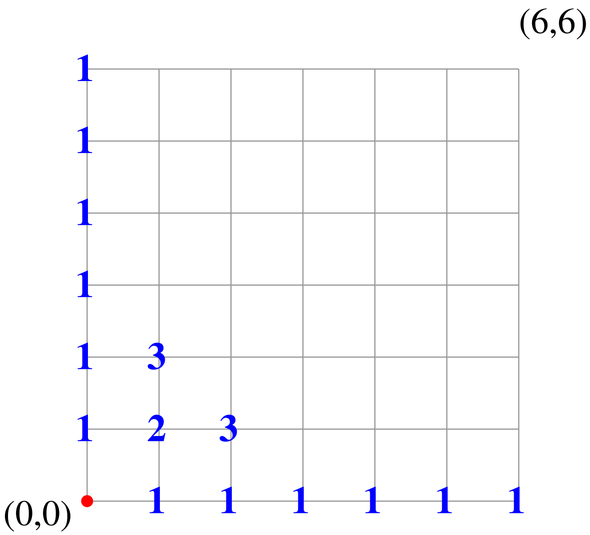
1
1 3
1 2
(0,0) 1 1 1 1 1 1



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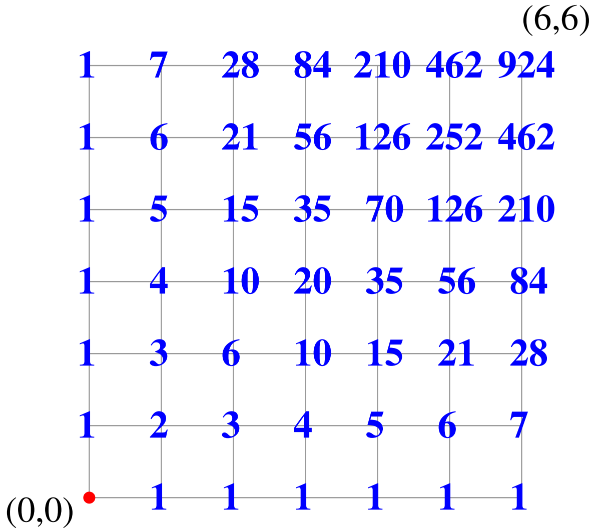

Answer to Problem 1.1: The number of N-E paths from $(0,0)$ to $(6,6)$ is 924.

In fact, looking carefully we see that this is Pascal's triangle, but lying on its side!

To see why this is true, every path can be written as a sequence of N (North) and E (East steps), e.g., $(NNENNNE)$. To get to the point (a,b), there are $a+b$ steps in total. So the sequence of steps has length $a+b$. Also, there must be a East steps and b North steps.

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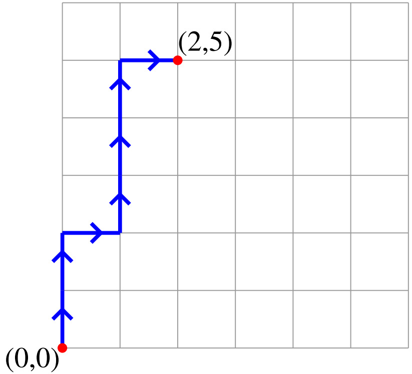
Specifying any path is the same as choosing where (among $a + b$ available positions) the a East steps occur. The number of ways to do this is equal to

$$\binom{a+b}{a} = \frac{(a+b)!}{a!b!}.$$

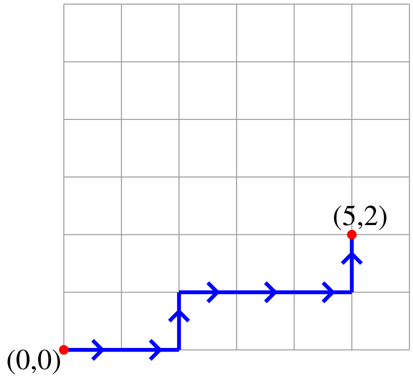
Notice the symmetry in the values of $f(a, b)$.

It may be summarised by saying $f(a, b) = f(b, a)$.

Can we give a reason for this? Consider a path from $(0, 0)$ to $(2, 5)$:



The path is composed of steps $(NNENNNE)$.
Changing any occurrence of N to E and vice-versa, we have the new path $(EENEEEN)$. This path goes from $(0, 0)$ to $(5, 2)$.



In fact it is the reflection in the line $y = x$ of the original path.
To every N-E path from $(0,0)$ to $(2,5)$ there corresponds a unique N-E path from $(0,0)$ to $(5,2)$.

Similarly, to every N-E path from $(0,0)$ to $(5,2)$ there corresponds a unique N-E path from $(0,0)$ to $(2,5)$.

This tells us that the number of N-E paths from $(0,0)$ to $(5,2)$ is the same as the number of N-E paths from $(0,0)$ to $(2,5)$. So $f(2,5) = f(5,2)$.

The same is true for any integers (a,b), so we have

$$f(a,b) \;=\; f(b,a).$$

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Exercise 1.18 Suppose we are working in 3-dimensional space instead of 2-dimensional space. If we begin at the point $(0, 0, 0)$ and may take steps North $(0, 1, 0)$, East $(1, 0, 0)$ and Up $(0, 0, 1)$, how many such paths are there from $(0, 0, 0)$ to $(4, 4, 4)$?

Exercise 1.19 How many N-E paths are there from $(0, 0)$ to $(9, 9)$ which do not go through the point $(5, 5)$?

Exercise 1.20 A spider has one sock and one shoe for each of its 8 legs. In how many different orders can the spider put on its socks and shoes, assuming that on each leg, the sock must be put on before the shoe?

Answer: $\dfrac{(16)!}{2^8}$.

Exercise 1.21 $2n$ tennis players participate in a tournament, where $n \geq 1$. In how many ways can they paired up to play simultaneously?

Answer: $\dfrac{(2n)!}{2^n n!}$.

Exercise 1.22 A shop has green, blue and red hats.

(i) How many ways are there to choose 10 hats, assuming at least 10 hats of each type are available?

(ii) What if there are only 3 red, 4 blue and 5 green hats left in the shop?

Exercise 1.23 (BMO Round 1, 2005-2006) Adrian teaches a class of six pairs of twins. He wishes to set up teams for a quiz, but wants to avoid putting any pair of twins into the same team. Subject to this condition:

(i) In how many ways can he split them into two teams of six?

(ii) In how many ways can he split them into three teams of four?