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MATHS SPARKS

Problem Solving Workshops

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What is Maths Sparks?

Maths Sparks is a series of problem solving workshops for senior-cycle secondary students, designed and run by students and staff of the UCD School of Mathematics and Statistics.

These workshops were created as a series of dynamic, interesting, fun and interactive learning experiences for students, which demonstrate the applications of mathematical concepts in a broad range of contexts.

We have found that students' participation in the Maths Sparks workshops has had a positive influence on their confidence in learning mathematics, in their motivation to learn mathematics and in their opinions on the usefulness of mathematics.

Information for Teachers

We hope this booklet will be a useful resource for teachers running Maths Clubs in their schools or for those teaching extra-curricular content, such as to Transition Year students. Each workshop in this booklet includes a lesson plan and resources which can be used in your classroom.

While the majority of content in these workshops is not taught at second level, the activities are designed to cultivate students' mathematical thinking and problem solving skills. The workshops are designed to encourage a more collaborative approach to teaching and learning mathematics, where students collaborate in pairs or groups in making sense of specific tasks. This is particularly relevant in the 'Engineering and Project Management' workshop, where your students will work in different teams to create the best proposal to win a contract.

Workshop topics range from cryptography to card-tricks and we hope you and your class will enjoy exploring these mathematical ideas.

Information for Third Level Institutions

'Maths Sparks' is a series of workshops delivered by students and staff in the UCD School of Mathematics and Statistics to secondary students from schools designated as DEIS (Delivering Equality of Opportunity in Schools).

Undergraduate and postgraduate students volunteer to take part in the programme and, over a number of weeks, design and peer-review workshops in small teams. Each university team presents their workshop to secondary school attendees and also facilitates student learning in other workshops throughout the series. Academic staff participate in workshops by mentoring student teams, by participating in the peer review process and by presenting their research work as a context for studying Mathematics beyond secondary school.

Organised in conjunction with the UCD Access and Lifelong Learning, this series of workshops is delivered free of charge to secondary students and provides them with experiences of learning mathematics outside of the classroom, opportunity to engage with university students and lecturers, and the opportunity to purposefully visit a university campus.

Feedback from our university volunteers has been very positive, with students reporting on the development of their communication, presentation and team-work skills. University volunteers have valued the opportunity to meet and engage with other students in different degree pathways and stages of study. Our university student volunteers have also valued the opportunity to work with academic staff in an extra-curricular capacity and many have continued to participate in outreach events following this volunteering experience.

Developing and participating in Maths Sparks has provided the UCD School of Mathematics and Statistics opportunity to cultivate student-staff relationships and to engage with communities outside of our institution.

We hope this booklet will be a useful resource for those who wish to establish a Maths Sparks programme or similar workshops in their institution.

Further Information

For further information please see: http://www.ucd.ie/mathstat/mathsparks/
If you would like to share feedback on the workshops or would like to develop a similar programme in your institution please email: aoibhinn.nishuilleabhain@ucd.ie

Background to Maths Sparks

In 2014–2015, UCD ran a programme to encourage students and staff to work together to enhance or improve an aspect of university or community life. This SPARC (Supporting Partnership and Realising Change) initiative awarded initial funding to a small number of projects where university students and staff would collaborate in realising their idea. A small team of staff and students from the UCD School of Mathematics and Statistics and the UCD Access and Lifelong Learning centre began planning a series of mathematics workshops for secondary students. While there are many outreach activities aimed at high–achieving students, research from higher education institutes across the UK has suggested that it is necessary to widen participation in mathematics outreach programmes to students of a more diverse range of social backgrounds (Cox and Bidgood, 2002). In the Irish context it has been found that students from DEIS schools are less likely to continue studying mathematics at higher level (Smyth et al., 2015) and, in an attempt to encourage more of these students to continue studying mathematics to their highest ability, students from DEIS schools around Dublin were invited to attend.

Following a few weeks of planning and trialling workshops, our eleven university volunteers presented and facilitated their series of problem solving workshops to over forty students from six different schools. A number of talks from mathematics graduates and lecturers were also included in the workshops. Feedback from secondary students was overwhelmingly positive, with the majority of students reporting an increase in confidence in their mathematical ability as a result of their participation. A number of students stated that they would consider studying mathematics after their Leaving Certificate (Ni Shúilleabháin & Cronin, 2015).

In 2015, the Maths Sparks team won funding from SFI Discover to expand the programme. More university students volunteered to participate in the programme and additional workshops were developed on mathematical topics spanning physics and engineering and incorporating more puzzles, games and activities (Badger et al., 2012). Partnering with the UCD Access and Lifelong Learning Centre, students from 11 different secondary schools were invited to attend. The series of workshops ran over six weeks, with over 70 students in attendance, and culminated in an awards ceremony for parents and teachers of participating students.

As a result of their participation in Maths Sparks, there was an increase in students' effectance motivation, confidence in learning mathematics and in recognising a usefulness in learning mathematics. Furthermore, our university students have gained valuable skills in presentation and team–work and have developed a sense of community through their collaborative work.

We plan to continue to run Maths Sparks in the UCD School of Mathematics and Statistics and hope to encourage more young people to study mathematics to their highest potential at secondary school. We also hope that participation in Maths Sparks will encourage students to consider continuing their study of Mathematics at third level.

Students and volunteers for Maths Sparks 2016

References

Badger, M., Sangwin, C., Ventura–Medina, E., & Thomas, C. (2012). A Guide to Puzzle–Based Learning in STEM Subjects. University of Birmingham: National HE STEM Programme.

Ni Shuilleabhain, A., & Cronin, A. (2015). Maths Sparks: Developing Community and Widening Participation. MSOR Connections, 14(1), 43–53.

Smyth, E., McCoy, S., & Kingston, G. (2015). Learning from the Evaluation of DEIS. Economic and Social Research Institute, Dublin.

An Undergraduate Perspective

Why might a university student volunteer for Maths Sparks?

We chose to get involved in Maths Sparks to gain valuable teaching experience whilst also taking the opportunity to make UCD a friendlier place for post–primary students to come. We have found that post–primary students are apprehensive about coming to the university, as they feel daunted by the sheer size and scale of everything compared to their own schools.

Maths Sparks provided us with the opportunity to explore what works well in communicating our knowledge and passion for mathematics, as well as planning outreach workshops or lessons for young people. It gave us the chance to improve our skills in thinking on the spot and working as part of a team, while also helping others to optimize their own learning.

It has been really rewarding to have such a positive influence on post–primary students' learning of Mathematics, with many telling us they are now more confident in their mathematical ability and more positive about attending university. From participating as workshop designers and facilitators, we have developed our communication and presentation skills and have also had the opportunity to collaborate on teams. In addition, volunteering with Maths Sparks has provided us with opportunity to work with lecturers and to meet UCD students from other stages and pathways within the School of Mathematics and Statistics, the College of Science, and across the university.

We would very much recommend other students to get involved in similar outreach initiatives during their studies.

Emily Lewanowski–Breen
Stage 2 Mathematics, Biology and Education

Christopher Kennedy
Stage 3 Applied & Computational Mathematics and Statistics

Workshop Content

Cryptography

Introduction

Cryptography is the art of producing or solving codes and has been used as a method of secure communication since as early as 1900 BC. Whilst Cryptography initially concerned communication and linguistics, it has become an incredibly important area of mathematics given its roots in number theory and its relevance to internet security. Text messages, emails and online banking, for example, are all secured with end–to–end encryption techniques to ensure the safety of our personal details. It is therefore evident that Cryptography still has modern day applications. One of most well–known examples of Cryptography in ancient times was the 'Caesar Cipher' which was first developed by Julius Caesar and reportedly used to communicate messages across the Roman Empire. The Caesar Cipher is considered one of the most simplistic forms of encryption, given that it uses a substitution technique whereby each letter is replaced by another further on in the alphabet. However, frequency analysis can be used to decipher such codes and therefore it is considered a relatively weak and unreliable method of encryption. That being said, the 'Vigenère Cipher', which is a variation of Caesar Cipher, is a more secure form of communication given that a keyword is used to encrypt the message and thus each letter has a different shift.

Aim of Workshop

This workshop will introduce students to the basic concepts of Cryptography including ciphers, decrypting codes and the use of prime numbers in Cryptography. Students will also be provided with the opportunity to create their own encrypted messages, which they can then give to their partner to solve. By the end of this workshop students will have a better understanding of how Cryptography works and its relevance to internet security.

Learning Outcomes

By the end of this workshop students will be able to:

- Encrypt and decrypt coded words using Caesar and VigenĐre ciphers
- Explain, in their own words, how 'modulus' works
- Produce their own ciphers
- Be familiar with historical decryption strategies

Materials and Resources

Alphabet line

Keywords

Cipher
A way of making a word or message secret by changing or rearranging the letters in the message.

Shift
A value, ***X***, which causes the letters to move ***X*** number of spaces up or down the alphabet line.

Cryptography: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
10 mins (00:10)	**Introduction to Workshop and Concept of Cryptography**	– Define what is meant by Cryptography (similar to workshop introduction) – Explain how modulus works (see **Appendix – Note 1**) – Mention applications of Cryptography such as internet banking, emailing, WhatsApp etc.
25 mins (00:30)	**Activity 1** **Caesar Ciphers**	– Brief introduction to the Caesar Cipher using a video clip (link included in **additional resources**) – Explain how to encrypt a word using an example on the board (see **Appendix – Note 2**) – **Activity Sheet 1**: In pairs, students attempt activity 1 using the alphabet line to guide them. Students are asked to pick a word and encrypt it using a particular shift value of the alphabet. Their partner must then attempt to break the code using the key shift.
5–10 mins (00:50)	**Activity 2** **Sentence Breaker**	– **Activity Sheet 2**: Students complete activity sheet 2 in pairs – Show students the encrypted sentence on **Activity Sheet 3** and ask them to try figure out how to solve it without given the key shift (looking at possible vowels etc.) – You can relate this activity back to the video clip shown at the start of the workshop
25 mins (01:15)	**Activity 3** **Vigenère Cipher**	– Introduce students to Vigenère Cipher and show how to encrypt using Vigenère by working through an example on the board (see **Appendix – Note 3**) – **Activity Sheet 3**: In pairs, students complete Activity Sheet 3 using Vigenère Cipher

Cryptography Workshop Appendix

Note 1: The Concept of 'modulus'

Modular arithmetic is a way of counting integers whereby numbers "wrap around" upon reaching a fixed quantity known as the **modulus**. For example, once we reach 12 on a clock, we start back at 1. The same applies in Cryptography whereby once the letter Z is reached, we go back to A.

Example 1: If we want to encrypt the letter 'T' using a shift of 8 on the alphabet line, then we will need to take the modulus into account. This is due to the fact that 'T' corresponds to the number 20 on the alphabet line and thus, by adding 8, we get a value of 28. However, there is no 28th letter in the alphabet so we get 26 remainder 2. The letter encoded by the number 2, in this case 'B', is therefore the encrypted letter for 'T'.

Note 2: How to Encrypt a Word using the Caesar Cipher

1. To encrypt a word using the Caesar Cipher, each pair of students will require two copies of the alphabet line (see appendix).
2. Students write down their word on a piece of paper and convert it to the relevant numbers on the alphabet line.
3. Students must then decide on a key shift **X** (a value between 1 and 26) to code their word and add this value to each of the numbers, taking the modulus into account where necessary.
4. Students then convert these new numbers back to letters. This is their encrypted word.
5. Students then hand their coded message and chosen shift to their partner so that they can decrypt it (by working backwards).

Example 1: Chris decides he wants to encrypt his name with a Caesar Cipher using a shift of 9.

1. What will his name look like after he encrypts it?

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

2. Convert the word "**Chris**" to the corresponding numbers using the alphabet line

c	h	r	i	s
3	8	18	9	19

3. Add 9 (the shift) to each number shown in blue above
4. Check the alphabet line to see what each of these new numbers now translates to

12	17	27	18	28
L	q	a	r	b

The encrypted message for "**Chris**" is thus "**Lqarb**"

Example 2: Alternatively, students can use their second alphabet line to find the new "secret" letters for their encrypted word. For example, if the shift is 3, students place the second alphabet number line in such a way that 1 now corresponds with 4 on the second alphabet line.
Notice how each letter has now shifted 3 places (which is the same as adding the shift to each number as in the example above). The letter "a", for example, has now become "d"– hence the word "apple" on the top line will correspond to the coded message "dssoh" on the bottom line. Using this method, students can directly read their coded message from the alphabet line.

Note: The overhang at the end of the alphabet line will, in theory, loop around such that x, y, z now become a, b, c respectively. This is the same idea behind the modulus.

			a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26			

Note 3: How to Encrypt a Word using the Vigenère Cipher

1. Write out the words you want to encrypt.
2. Find the corresponding number for each letter of your message from the number line and also the corresponding number for each letter of the keyword.
3. Write and match up the keyword with your message.
4. Replace the letters with the numbers.
5. Add the numbers together.
6. Locate the numbers on the alphabet line to give you your encrypted message.

Example 1: Maria wants to encrypt the message "See you later" using a VigenĐre Cipher and the keyword "maths". What will her message look like after she encrypts it?

1. Message to encrypt = **See you later**

   Keyword = **maths**

2. Find the corresponding number for each letter

s	e	e	y	o	u	l	a	t	e	r
19	5	5	25	15	21	12	1	20	5	18

m	a	t	h	s
13	1	20	8	19

3. Match the keyword with the message by repeating it until all letters of the message are accounted for

s	e	e	y	o	u	l	a	t	e	r
m	a	t	h	s	m	a	t	h	s	m

4. Now replace the letters above with the numbers

19	5	5	25	15	21	12	1	20	5	18
13	1	20	8	19	13	1	20	8	19	13

5. Add the numbers on the top with those on the bottom

32	6	25	33	34	34	13	21	28	24	31

6. Locate the numbers (in yellow) on the number line to give the encrypted message, taking the modulus into account

f	f	y	g	h	h	m	u	b	x	e

Encrypted message is thus: **ffy ghh mubxe**

Sources and Additional Resources

https://www.youtube.com/watch?v=sMOZf4GN3oc (Caesar Cipher video link)

http://www.furthermaths.org.uk/files/Encryption.pdf

http://practicalcryptography.com/ciphers

Activity 1 – Caesar's Cipher

You can use the alphabet line to help you encrypt or decrypt the messages.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

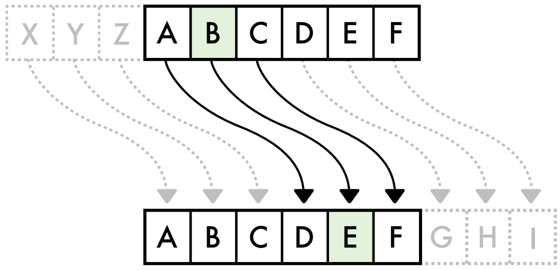
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z

1. **Daniel decides he wants to encrypt his name with a Caesar Cipher using a shift of 10.**

   Which of the following represents his encrypted name?

   - [ ] a. nkxsqv
   - [ ] b. ncpkgn
   - [ ] c. nkxsov
   - [ ] d. none of the above



2. **Jessica decides she wants to encrypt her name with a Caesar Cipher using a shift of 22.**

   Which of the following represents her encrypted name?

   - [ ] a. faggwqo
   - [ ] b. faoqgay
   - [ ] c. faooeyw
   - [ ] d. none of the above

3. **Thomas wants to encrypt his name with a Caesar Cipher using a shift of 15.**

   What will his name look like after he encrypts it?

4. **Ishaan is expecting to receive an encrypted message with the name of one of his classmates. He knows the message will be encrypted with a Caesar cipher using a shift of 10. When he returns to his desk, he finds a piece of paper with the following message on it:**

   l b k x n y x

   Help Ishaan crack the code by writing the decrypted message in the answer box

5 **Tiffany receives an encrypted message with the name of one of her classmates. She knows the message is encrypted with a Caesar cipher using a shift of 11. The piece of paper has the following code on it:**

x t n s l p w

Help Tiffany crack the code by writing the decrypted message in the answer box

6. **Gabriela decides she wants to encrypt her name with a Caesar Cipher using a shift of 17.**

What will her name look like after she encrypts it?

Additional Questions

1. **Using a shift of 7, crack what Caesar is thinking.**

2. **In pairs, come up with a word, encrypt it using Caesar's Cipher and then give it to your partner to decrypt. The key shift must also be given to your partner.**

Activity 2– Sentence Breaker

Figure out the shift for an encrypted sentence.

H a p   i a   p w g a   w   o a h b e a

UCD Maths Sparks Facilitators, 2015

Activity 3 – Vigenère's Cipher

1. **Emily wants to encode the following cryptic message:**

Uptown funk

In an effort to increase security, **Emily** decides to encrypt it with a VigenÐre cipher using her own name as the keyword. What will the secret message look like once it is encrypted? Use extra paper to figure out the encryption and write your coded answer in the box below.

2 **Using the word Vigenère as the shift, encrypt the sentence below and place in the speech bubble to code what Vigenère is saying.**

I'm smarter than Caesar

Game Theory

Introduction

Game Theory is an important field of mathematics which concerns the analysis of game strategies and is applicable to a wide range of disciplines, including economics and politics. Essentially, Game Theory aims to identify complete solutions to games or situations which can outline how an individual can place themselves in the best possible (winning) position. However, this can often prove challenging depending on the simplicity of the game or situation. For example, if we compare the games Connect 4 and Monopoly, these would involve very different solutions.

There are two distinct classifications of Game Theory: Combinatorial Game Theory and Classical Game Theory. Combinatorial Game Theory focuses on the study of two–player games whereby both players know all of the rules and take alternate moves. Furthermore, there are no 'chance' elements to such games and each player can see the moves that have been previously made. In contrast, Classical Game Theory involves players moving or strategizing simultaneously and is characterised by concealed information and elements of chance.

Aim of the Workshop

The aim of this workshop is to introduce students to the concept of Game Theory and the application of **backwards induction** to a combinatorial game to solve various problems. Students use their knowledge of probability and statistics in an attempt to find a solution to a number of games including 'NIM' and 'Guess the Average' game.

Learning objectives

By the end of this workshop students will be able to:

- Discuss the relevance of Game Theory to applications in real life
- Provide a description (in their own words) of backwards induction
- Assign winning and losing positions in the NIM game
- Describe what is meant by 'optimal strategy'

Materials and resources

21 sticks/counters for each pair of students, sheets of paper.

Keywords

Optimal Strategy
A sequence of moves which leads to the best outcome of the game or situation.

Backward induction
A method used to solve sequential games whereby a player works backwards in order to determine the optimal strategy of the game.

Dominant Strategy
A strategy is considered dominant if it earns a player a better payoff in the game, regardless of the other players' actions.

Nash Equilibrium
A set of techniques in which no player can benefit from a change in their strategy as long as all other players' strategies remain unchanged. It is also relevant to economics and was first developed by John Forbes Nash Jr, an American mathematician whose extraordinary life inspired the storyline to the film "A Beautiful Mind".

Game Theory: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Introduction to Game Theory**	– Discuss 'Game Theory' as relevant to real life
10 mins (00:15)	**Combinatorial Game Definition and Traits**	– Discuss the features of combinatorial games (see **Appendix – Note 1**) – Task: ask students if a selection of games are/are not combinatorial and why (see **Appendix – Note 2** for list) – Task: ask students to list and justify extra examples of combinatorial games
10 mins (00:25)	**Introduction to NIM Game**	– Explain the rules of the NIM game (see **Appendix – Note 3**)
10 mins (00:35)	**Optimal strategy**	– Define what is meant by 'optimal strategy' – Ask students if they can come up with an optimal strategy to win the NIM game – Task: Divide students into pairs and ask them to play the game at least 3 times (can take 1, 2 or 3 sticks) – Ask if anyone has come up with a winning theory – Class discussion of ideas
10 mins (00:45)	**Backwards induction**	– Task: Introduce 'backwards induction' whereby students attempt the same game with 4 sticks – Class discussion on solution – Task: Using backwards induction, students play with 21 sticks again and try to identify a pattern or strategy
15 mins (01:00)	**Change the NIM values: can take 1, 3, or 4 sticks at each turn**	– Students play the new game and attempt to come up with an optimal strategy using the same idea of backwards induction as before

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 –10 mins (01:10)	**Introduction to the Pirates Puzzle**	– Introduce the Pirate Puzzle (see **Activity Sheet 1**) – Ask five volunteers to come up and demonstrate possible scenarios for the five pirates puzzle – Suggest approach and layout to the problem (see Appendix – **Note 5**)
10–15 mins (01:25)	**Doing the Puzzle**	– Task: **Activity Sheet 2** In pairs, students work together using backwards induction (starting with 2 pirates and working up to 5 pirates) to determine the optimal suggestion for the eldest pirate – Whole class discussion on strategies and solution (as noted in Appendix – **Note 4**)
5 mins (01:30)	**John Nash**	– Briefly discuss John Nash before introducing the Nash equilibrium – (option to play 'A Beautiful Mind' clip (link included in Additional Resources))
10 – 15 mins (1:45)	**Guess the average game**	– Choose 10 volunteers for this activity and explain how the game works (see Appendix – **Note 5**)
5 mins (01:50)	**Discussion and link to Nash equilibrium**	– Explain how the numbers decrease after each iteration due to everyone being aware of the game and decreasing their guess accordingly. The Nash equilibrium is zero
5 mins (01:55)	**Conclusion of Workshop**	– Encourage students to write down three things they have learned from the workshop and encourage students to independently follow up on any questions or queries they might have

Game Theory – Workshop Appendix

Note 1: Criteria for Combinatorial Games

1. Two players who take alternate moves.
2. The rules of the game specify the legal moves for both players to and from each position.
3. There are no elements of chance (e.g. using a die)
4. The game eventually comes to an end.
5. There are no draws and the winner is determined by the player who makes the final move.

Note 2: Examples of Combinatorial games

Combinatorial games: Chess, Checkers, Connect 4, NIM, Go

Non-combinatorial games: Monopoly (has an element of chance), Rock, Paper, Scissors (players take turns simultaneously), soccer (has no legal moves), Xs and Os (can result in a draw)

Note 3: Rules for NIM Game

1. Each pair is provided with 21 sticks which are placed into a row on the table.
2. Students take alternate turns removing sticks from the table
3. Students take a defined number of sticks during their turn (e.g. 1, 2 or 3).
4. The student who picks up the last stick wins the game

(In this game there is a strictly dominant strategy. By this we mean that once the strategy is known to one of the players, there is no way to beat this player. Hence, this optimal strategy strictly dominates any strategy used by the opposing player.)

Solution – (picking up 1, 2, or 3 sticks)

Students can derive the solution by using the blank Activity Sheet 1 and highlighting winning versus losing positions or turns. In this case, all multiples of 4 (4n) are losing positions. Since 21 is not a multiple of 4, the winning strategy is to be the first player to move and to pick up only one stick.

Solution – (picking up 1, 3, or 4 sticks)

Students can derive the solution by using the blank Activity Sheet 1 and highlighting winning versus losing positions or turns. This time the losing positions are multiples of 7 (7n) or multiples of 7 plus 2 (7n + 2). This time, since 21 is a multiple of 7, the winning strategy is to be the second player to move.

Note 4: Solving the Pirate Puzzle

1. Label the pirates A, B, C, D and E for convenience, with A being the eldest.
2. Keep in mind that the eldest pirate wants to pay the other pirates the least amount necessary.
3. Remember the most important thing to each pirate is survival and then coins.
4. Work backwards (as in the NIM game) to decide the best outcome so that the eldest pirate stays alive and gets as many coins as many coins as possible.

	Pirate				
	A	B	C	D	E
1	-	-	-	-	100
2	-	-	-	100	0
3	-	-	99	0	1
4	-	99	0	1	0
5	98	0	1	0	1

Solution:
(You may wish to not provide this suggested table to students during the workshop)

One Pirate: When there is only one pirate (in this case Pirate E), he will get all 100 coins.

Two Pirates: In the case of two pirates D and E, Pirate D will propose to split the coins 100 : 0. His vote (50%) is enough to secure this deal and hence he is not thrown overboard.

Three Pirates: Pirate C (being the eldest of C, D and E) will suggest to split the coins 99 : 0 : 1. Pirate D will vote against this. However, pirate E will accept the offer of getting just 1 coin as he is aware that if he rejects the deal there will be only two pirates left and he will therefore get nothing.

Four Pirates: In the case of four pirates (B, C, D and E), Pirate B will choose to divide the coins 99 : 0 : 1 : 0. By the same reasoning as before, Pirate D will support this offer as he knows that if he rejects it there will be three pirates left and he will then get nothing. Pirate C will always vote against the offer because he knows that if his vote wins, Pirate B will be thrown overboard and he will therefore get 99 coins (as previous in the example). Pirate B will therefore not waste a coin on Pirate C. Likewise Pirate E will also vote against the proposal as he knows that if Pirate B is thrown overboard, he will get 1 coin in the following round (three pirates). However B and D still have 50% of the vote.

Five Pirates: Pirate A will split the coins 98 : 0 : 1 : 0 : 1. By offering 1 coin to Pirate C and E (who would otherwise get nothing) he secures the deal.

Note 5: Guess the Average Game

1. 10 volunteers are chosen. Each volunteer writes a number between 0–100 on a piece of paper but keeps it covered.
2. Volunteers asked to take a guess at what they think half the average of all ten guesses will be.
3. Students think and justify their reasoning to the class. (For example, all answers above 50 can be automatically ruled out. As the highest possible average is 100, and half of that is 50. Now, if everyone playing the game thinks this way, the highest possible average of the guesses is 50. So, half this value would be 25. This means every value above 25 can also be ruled out etc).
4. Calculate the average of the 10 guesses

For example:

If the 10 guesses are: 35, 50, 18, 65, 42, 31, 24, 12, 49, 14

The average is therefore: 35 + 50 + 18 + 65 + 42 + 31 + 24 + 12 + 49 + 14 = 34

So, half the average is 17 and the nearest guess to 17 will win the game.

Sources and Additional Resources

http://mathworld.wolfram.com/GameTheory.html

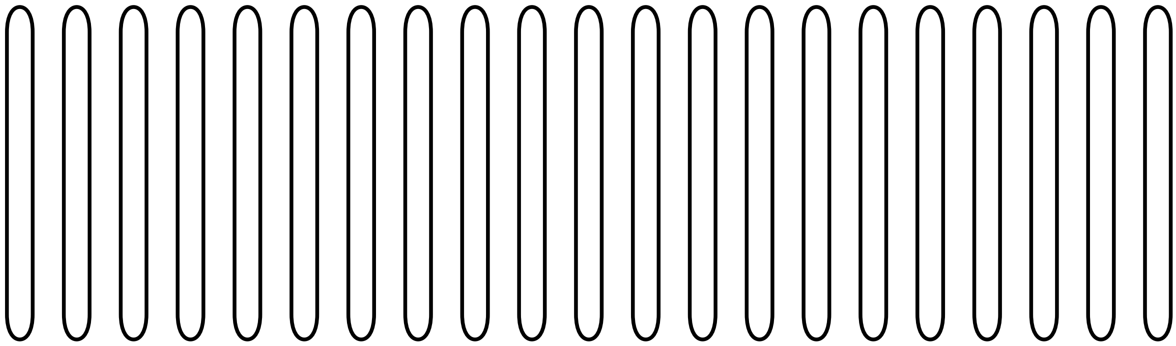
http://www.mathsisfun.com/puzzles/5-pirates.html

https://www.youtube.com/watch?v=LJS7Igvk6ZM (Beautiful Mind Clip)

Freakonomics Podcast: How to Win Games and Beat People

'How to Win Games and Beat People' – Book by Tom Wipple

Game Theory: Activity Sheet 1



Game Theory: Activity Sheet 2

Pirate Puzzle

5 pirates of different ages have a treasure of 100 gold coins.

On their ship, they decide to split the coins using this scheme:

- The oldest pirate always proposes how to share the coins, and ALL pirates (including the oldest) vote for or against it.
- If **50% or more** of the pirates vote for it, then the coins will be shared that way. **Otherwise**, the pirate proposing the scheme will be **thrown overboard**, and the process is repeated with the remaining pirates.
- If a pirate would get the same number of coins if he voted for or against a proposal, he will vote against so that the pirate who proposed the plan will be thrown overboard
- Assuming that all 5 pirates are intelligent, rational, greedy, and do not wish to die, (and are rather good at maths for pirates) what will happen?

Let **A** be the eldest pirate and **E** the youngest. Starting with just 1 pirate and working up to 5, can you fill out the table below to see how many coins the pirates will get? The first one has been completed for you.

	Pirate				
	A	B	C	D	E
1	-	-	-	-	100
2					
3					
4					
5					

Base Systems

Introduction

The decimal system is the most widely used numerical system in society and is commonly referred to as 'base 10', given that the basic units increase by powers of ten. It is believed that this system of counting developed due to the presence of 10 digits on our hands. However, various ancient cultures used different numerical systems which still have a wide range of applications in today's world. For example, the Babylonian number system, which used 60 as its base, is still used for measuring angles, time and even geographic coordinates. Similarly, the Mayan system used base 20 and we often come across references to it in different languages – for example, "Four score and seven years ago…" in English or "quatre-vingts" ("four twenties") in French. Base systems are also important in the information technology industry. For instance, binary (base 2) serves as the basis of computer software given that it only depends on two numbers or states: 0 and 1. However, many programmers have also made use of the octal (base 8) and hexadecimal (base 16) numeric base systems to enhance data memory and storage.

Aim of workshop

The aim of this workshop is to introduce students to the fundamentals of base systems, with emphasis on binary and hexadecimal, whilst also providing insight into the specific applications of these numerical systems. This workshop should also enable students to gain a deeper understanding of how the decimal system developed, whilst also exploring different ways of counting.

Learning Outcomes

By the end of this workshop students will be able to:

- Explain, in their own words, what is meant by a base system and why we developed to using the base 10 or decimal system in everyday life
- Recognise the features of binary and hexadecimal (number of symbols, applications etc.)
- Convert a decimal number into binary or hexadecimal and vice versa
- Apply similar methods to convert a decimal number into any other base system (e.g. base 3, base 8 etc.)

Materials and Resources

(Both optional) Laminated base systems templates, whiteboard markers.

Key Words

Base system
A way of expressing numbers, using digits or other symbols, in a consistent manner.

Binary
A system of counting which has '2' as its base rather than '10'. It is also commonly referred to as 'base 2'.

Hexadecimal
A system of numerical notation that has '16' as its base (once we reach the number 16, we start back at 1 again). It is also known as 'base 16' and uses the digits 0 to 9 and the letters 'a' to 'f'.

Base Systems: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Introduction to workshop and revision of place values**	– Introduce students to the Babylonian and Mayan base systems (see **Introduction**) – Revise decimal place values with students (units, tens, hundreds, tenths, hundredths, etc.) – You may like to ask students why they think the decimal system is most common (e.g. 10 fingers)
5 mins (00:10)	**Activity 1– Place Values**	– Revise 10^0 (10 to the power of zero) – **Activity Sheet 1:** Divide students into pairs and ask them to complete Activity Sheet 1 as revision
5 mins (00:15)	**Base Systems Template**	– Introduce students to the base systems template (included below) – Demonstrate how to use the template using an example (see **Appendix – Note 1**)
10 mins (00:35)	**Introduction to Binary**	– Introduce students to binary (base 2) and its application in computer technology – Mention the number symbols in binary (e.g. 0 & 1) – Ask students "What do the symbols on a binary light switch mean?" (0 = off and 1 = on) – Show students how to convert a number from decimal into binary and vice versa using the template (see **Appendix – Note 2**)
10 mins (00:45)	**Activity 3 – Binary**	– **Activity Sheet 3**: In pairs, students attempt Activity Sheet 3 and should be encouraged to explain their thinking to one another – (Students may use a laminated template and whiteboard marker to help them. Alternatively, the template can be printed and filled in using a pencil)

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
10 mins (00:50)	**Introduction to Hexadecimal**	– Introduce students to hexadecimal (base 16) – Ask students "How many symbols do you think are required for hexadecimal?" and discuss suggestions for additional symbols that could be used (see **Appendix – Note 3**) – Discuss examples using the template – What would 16 be in hexadecimal? (This can be constructed using the template to find the answer of 10)
5 mins (01:00)	**Activity 4 – Hexadecimal**	– **Activity sheet 4:** In pairs, students attempt activity sheet 4 – Discussion of solutions with whole class

Base Systems – Workshop Appendix

Note 1: How to use the Template

Example 1: When working with place values it is often helpful to use a template. In the following example, we are using base 10:

What **number** is given here when the following three numbers are added?

- 7 x 102
- 3 x 100
- 4 x 103

1. Write the 'base number' into each of the boxes on the first row – in this case we are using base 10. Note: the indices are already included in the template.
2. Fill the given values into the second row, ensuring that they are in the correct column.
3. For each column, multiply the first and second row together and record your answer in the third row (see figure 1 below)
4. Add across all the values in the third row to get the answer i.e. 4000 + 700 + 0 + 3 = 4703

Figure 1: Diagram showing how to use base systems template for decimal values

10^5	10^4	10^3	10^2	10^1	10^0	
		4	7	0	3	
		4×10^3 $=4000$	7×10^2 $=700$	0×10^1 $=0$	3×10^0 $=3$	Total 4703

Note 2: How to Convert a Number from Binary to Decimal and Vice Versa

Example 1: When converting a number from binary to decimal it is often helpful to use a template. In the following example, we are converting the binary number "1001" into decimal.

1. Fill in the 'base number' into each of the boxes on the top row (in this case we are using base 2)
2. Write the binary number that you wish to convert to decimal into the second row, making sure to separate the digits (e.g. 1001)
3. Multiply the top two rows together to get the third row e.g. 20 x 1 = 1, 21 x 0 = 0 etc.
4. Add across each of the boxes in the final row to get the converted decimal answer e.g. 8 + 0 + 0 + 1 = 9

5 The resulting decimal answer is therefore 9

Figure 2: Diagram showing how to convert from binary to decimal using the template

2^5	2^4	2^3	2^2	2^1	2^0	
		1	0	0	1	
		8	0	0	1	Total = 9

Example 2: When converting from decimal to binary, we need to work backwards and think about what 'base 2' values make up the decimal number when added together. In the following example, we are converting the decimal number '6' into binary.

1. Fill in the 'base number' into each of the boxes on the top row (in this case we are going to convert to base 2)
2. Ask yourself "what base 2 values make up the given decimal number?"

   E.g. In the case of the decimal number 6, we know that anything above 23 (=8) is too large. We therefore look at the remaining base 2 values in the template:

   22 = (4) 21 = (2) 20 = (1)

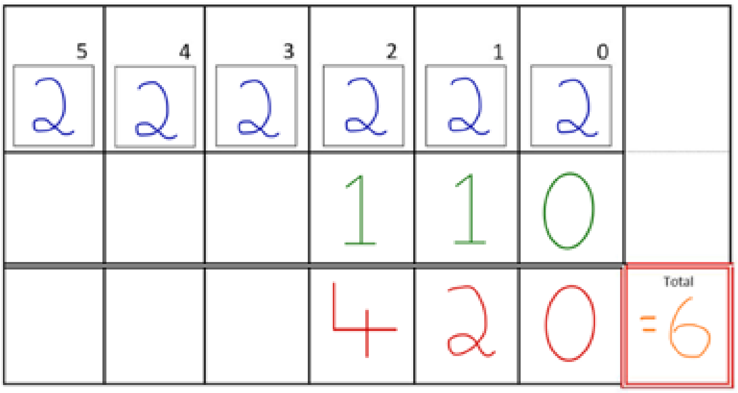
   By only taking one or none of the above values, how can we make 6? Trial and error is the best way forward.

   Well, we could take one 22, one 21 and zero 20 which would equal 6 when added together i.e. 4 + 2 + 0 = 6. We can fill this into our template (as shown below).

   Note: it is important to include the 0 under 20 given that the binary number '110' is very different to the binary number '11'.
3. The answer can be checked by multiplying the top row by the second row and adding to see if we get 6.

4. The decimal number 6 is thus 110 in binary.

Figure 3: Diagram showing how to convert from decimal to binary using the template

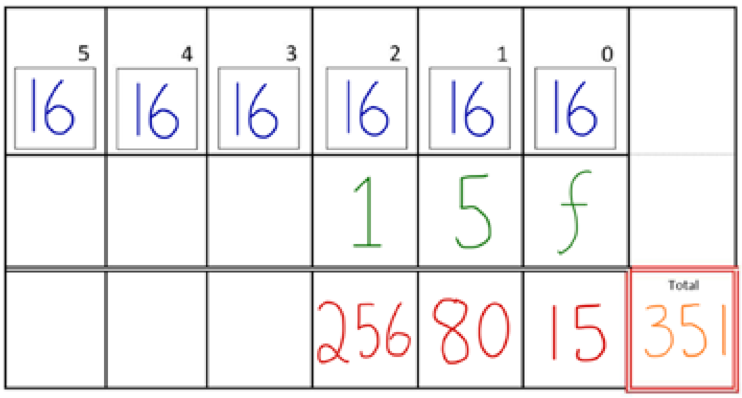


Hexadecimal uses 16 distinct symbols, however, we are only familiar with our distinct numbers 0 to 9. Therefore, additional symbols, in this case a to f, are used to represent the decimal values 10 to 15.

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f

When converting a number from hexadecimal to decimal we can use the same method as shown in Note 2, Example 1. For instance, the hexadecimal value '15f' would be 351 in decimal (see figure 4).

Figure 4: Diagram showing how to convert from hexadecimal to decimal using the template



Sources and Additional Resources

http://www.binaryhexconverter.com/decimal–to–binary–converter (Decimal to Binary Converter)

http://www.binaryhexconverter.com/decimal–to–hex–converter (Decimal to Hexadecimal Converter)

http://www.purplemath.com/modules/numbbase.htm

http://www.storyofmathematics.com/mathematicians.html

Base Systems Activity 1

Place Values

1. What number is given here when added?

- 7 hundreds
- 3 units
- 4 thousands

2. What number is given here when added?

- 5 units
- 1 ten thousand
- 3 hundreds

3. What number is given here when added?

- 9 tens
- 2 thousands

4. What number is given here when added?

- 5 x 101
- 9 x 104
- 1 x 102

5. What number is given here when added?

- 6 x 100
- 7 x 103
- 2 x 101

6. What number is given here when added?

- 4 x 105
- 8 x 100
- 7 x 101

7. Describe, in your own words, how changing the powers of 10 changes the values of the number.

8. Do we use base 10 for all of our everyday numerical dealings? Describe examples where we use other bases.

Base Systems Activity 2

Binary (Base 2)

1. **The following numbers are written in binary form (base 2). Can you convert them into decimal (base 10)? You may use the template provided to help you.**

(a) 111

(b) 11011

(c) 10110

Base Systems Activity 2

2. **If we work the other way, can you translate these decimal numbers into binary? You may use the template provided to help you.**

(a) 9

(a) 24

(a) 37

Base Systems: Activity Sheet 2

3. In base 10, **Lucy** is 16 years old. In base 2, **Patrick** is 1101 years old? Who is the eldest?

4. Come up with some challenging number conversions for the person beside you.

Base Systems Activity 3

Hexadecimal (Base 16)

1. Can you calculate your **age** in hexadecimal?

2. What would today's **date** be in hexadecimal?

Base Systems – Blank Template

5	4	3	2	1	0	
						Total

27 Card Trick & Base 3

Introduction

(Note: It might be useful to do this workshop after the "base systems" workshop)
While there exist many card tricks that rely on various different methods to work, such as sleight of hand and tricks of the senses, few people realise that there are numerous tricks that can be performed using seemingly unrelated areas of mathematics. The 27 card trick is one such trick. Its beauty lies not in the performance. but in the mathematics behind it. It relies on writing numbers in "base 3" and depends on how we arrange the cards into piles.

Aim of the Workshop

The aim of this workshop is to (re–)introduce students to the fundamentals of base systems with emphasis on ternary (base 3) numbers, whilst also providing the opportunity to apply this knowledge to a seemingly unrelated area: Magic Tricks. It is hoped that this workshop will enable students to gain a deeper understanding as to how the decimal system likely developed, whilst also exploring different and entertaining ways of counting.

Learning Outcomes

By the end of this workshop students should be able to:

- Explain, in their own words, what is meant by a base system and why we developed to using the base 10 or decimal system in everyday life
- Recognise the reasons why other base systems might exist
- Convert a decimal number into ternary (i.e. base 3)
- Perform the 27 card trick while understanding the mathematical background.

Materials and Resources

Each pair of students will require: a deck of 27 cards (split a pack of 52 into two suits for each student and give them one joker card each), activity sheets and whiteboard marker.

Key Words

Base system
A way of expressing numbers, using digits or other symbols, in a consistent manner.

Ternary / Base 3
A system of counting which has '3' as its base counting value rather than '10'. It is therefore also commonly referred to as 'base 3'.

27 Card Trick & Base 3: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Introduction and demonstrate the card trick**	– Demonstrate the trick with one student as a volunteer (see **Appendix – Note 1** for guide) (don't give away the secret of the trick just yet)
10 mins (00:15)	**Counting systems – base 10**	– Introduce counting systems using base 10 (decimal system) [this may be revision, depending on if you have done the Base Systems Workshop] "Why do we think that the decimal system is the one we use in everyday life?"(Main reason being, we have 10 digits) – Explore one example of splitting a number into units, tens, hundreds, thousands etc. and write these as powers of 10 (see **Appendix – Note 1** for example) – Explain the importance of having 10 as the "base" of all the powers – Ask students for their insight to "Why can we write 1 = 100?" – Students can individually attempt **Activity Sheet 1** – While the students are doing Activity Sheet 1 project the blank table from the workshop in the resources onto the whiteboard or draw it out on the board.
10 mins (00:25)	**Mention other counting systems** **Extend to base 3**	– Mention other common bases that can be used such as binary and hexadecimal and give insight into their applications in the world of technology (see bases workshop) – Introduce the concept of base 3 – Explore an example of writing a number in base 3 on the board – Split the students into groups of two or three and give each group one copy of **Activity Sheet 2** from 'A' to 'I' (Alternatively give each group 3 numbers from 0 – 26 to calculate in base 3 and use **Activity Sheet 2**)

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:30)	**Go through the solutions to activity sheet 2 on the board**	– One student from each group should go to the board and fill in the 3 numbers they calculated in base 3. – Once all groups have gone, ask students to spot any mistakes in the table and fix accordingly. – Ask students if they can see any patterns in the table? – Focus their attention towards the patterns in the columns (the patterns are shown in colour in **Activity Sheet 2 – Card Trick Guide**)
5 mins (00:35)	**Explain how base 3 relates to the trick**	– Link the card trick to the table on the board and explain that to get the important card to a certain position (i.e. the favourite number given by a volunteer), we need to look at how many cards we need above it (see **Appendix – Note 2: teacher's guide**) – **KEY IDEA:** this means we have to subtract 1 from the number given. Relate this back to the demonstration at the start of the lesson. – Explain the idea of labelling • The top pile as the 0th pile, • The middle pile as the 1st pile • And bottom pile as the 2nd pile – Ask "Where have we seen 0s 1s and 2s already?" (On the table on the board)
10 mins (00:45)	**Perform the trick in "annotated mode"**	– Go through the trick again, slowly explaining how you pick up the piles after the volunteer points to the pile with their card in it – **KEY IDEA:** 1st deal — look at the 1s (units) column 2nd deal — look at the 3s column 3rd deal — look at the 9s column – Ask "Where would you put the pile for [different chosen numbers]?"
15 mins (01:00)	**Let the students try the trick themselves**	– In pairs get the students to try the trick on each other – Refer back to the table to split up the number and remember the **TWO KEY IDEAS**: • Always subtract 1 from the number they say at the start • When picking up the piles look at the units first, then the 3s and lastly the 9s – Students might wish to use print outs of the card trick table as necessary (see **Appendix – Note 3**)

27 Card Trick Appendix

Note 1 – Example of base 10 calculations

When we write numbers in the decimal system we are really writing numbers as a sum of units, tens, hundreds, thousands and so on:

E.g.

$$7{,}063 = 7000 + 0 + 60 + 3$$

$$= (7 \times 1000) + (0 \times 100) + (6 \times 10) + (3 \times 1)$$

And by using our knowledge of powers we can write the above as follows

$$= (7 \times 10^3) + (0 \times 10^3) + (6 \times 10^3) + (3 \times 10^3) +$$

So we can see in each bracket above we have our digit (0,1,2,3,4,5,6,7,8,9) multiplied by 10^n where $n \in \mathbb{N}$. So we can write any number as a sum of powers of 10. The fact that we use 10 as our "base" for the power terms is why we call the system "base 10"

Note 2 – How to perform the trick

The goal of the trick is to get the volunteer's chosen card to the position given by their favourite number, without ever knowing what their card was.

- Count out 27 cards from the full deck (or take 2 suits and a joker card [13+13+1 = 27])
- Ask for a volunteer
- Allow the volunteer to choose one of the 27 cards and ask them to memorise it (or show it to the class)
- Ask them to place the card anywhere in the remaining 26 and shuffle the cards
- Begin dealing the 27 cards face up into **3 equal piles** (this will be done 3 times)
- Ask the volunteer to keep an eye out for which pile their card goes into, but not to let you know when they see their card (it must be kept secret until the end!).
- While dealing ask the student "If you had to pick a number between 1 and 27 which would you choose?"
- When they give you the number immediately subtract 1 from it (to yourself) as we need this many cards on top of their chosen card (e.g. if they say '12' – to get the 12th position we need 11 cards on top of that)
- Convert this number to base 3 in your head (this will take practice)

$$11 = (1 \times 9) + (0 \times 3) + (2 \times 1) = (1 \times 32) + (0 \times 31) + (2 \times 30)$$

- Once the cards are dealt, ask the volunteer which pile their card went into. You must now decide where to place this pile for your next deal

– Where you should put this pile of cards is decided by the number of units calculated for each base. This first time you are focusing on and the rule is:

If 0, the pile goes to the **top**

If 1, the pile goes to the **middle**,

If 2, the pile goes to the **bottom**

- E.g. In our earlier example, '11' has **2** units of 3^0 so the pile they point to must go on the **bottom** (the order of the other piles does not matter)

– Deal out the cards again into 3 equal piles and get the student to point to the pile their card went into

– The pile they point to after the second deal must now go to the position (0, 1 or 2) decided by the number of units calculated for each base. So this time you are focusing on 3^1.

- E.g. In our example 11 had **0** units of 3^1 so the pile they point to goes on top of the other **two** piles (the order of the other piles does not matter)

– Deal the card out one last time and ask which pile their card is in and this time place it on top, middle or bottom depending on the number of units calculated for each base. This time you are focusing on the 3^2.

- E.g. In our example there was **1** nine, so the pile they point to goes back in the **middle**

– The trick is now complete so ask them to tell you what card they chose.

– Then ask them to repeat their chosen number between 1 and 27

– Count out that many cards and, if all goes well, the position they chose should be their chosen card!

– As another example, let's imagine that the volunteer says the number 13

- **Step 1:** Subtract 1 from this number, giving us 12
- **Step 2:** Split the number up into base 3
  - 12 contains one 9, one 3 and zero units so we can write it as:
- **Step 3:** When dealing the piles out and asking them to point to the pile their card is in, when putting the piles back together
  - 1st deal look at the **units,** 12 has zero – put their pile back on **top** (0th)
  - 2nd deal look at **threes**, 12 has **one** – put their pile back in the **middle** (1st)
  - 3rd deal look at **nines,** 12 has **one** – put their pile back in the **middle** (2nd)
- Now their chosen card will be in the ***13th position***

– For a video demonstration and further discussion, view the following: https://www.youtube.com/watch?v=l7lP9y7Bb5g

Note 3 – Activity Sheet 2 – Card Trick Table

NUMBER CHOSEN	NUMBER – 1	3^0	3^1	3^2
1	0	0	0	0
2	1	1	0	0
3	2	2	0	0
4	3	0	1	0
5	4	1	1	0
6	5	2	1	0
7	6	0	2	0
8	7	1	2	0
9	8	2	2	0
10	9	0	0	1
11	10	1	0	1
12	11	2	0	1
13	12	0	1	1
14	13	1	1	1
15	14	2	1	1
16	15	0	2	1
17	16	1	2	1
18	17	2	2	1
19	18	0	0	2
20	19	1	0	2
21	20	2	0	2
22	21	0	1	2
23	22	1	1	2
24	23	2	1	2
25	24	0	2	2
26	25	1	2	2
27	26	2	2	2

Remember: 0 = top pile   1 = middle pile   2 = bottom pile

27 Card Trick – Activity 1

$$7{,}603 = 7 \times 10^3 + 6 \times 10^2 + 0 \times 10^1 + 3 \times 10^0$$

Can you write the following numbers in the base 10 format shown in the example above?

1) 78021

2) 26

3) 36.21

Hint for question 3:

$$0.1 = \frac{1}{10} = 10^{-1}$$

$$0.7 = \frac{7}{10} = 7 \times 10^{-1}$$

27 Card Trick – Activity 2 (A)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?
Write them like the first line in the example above (shown in blue)

1) 0 ☐

2) 10 ☐

3) 18 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (B)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?
Write them like the first line in the example above (shown in blue)

1) 4 ☐

2) 16 ☐

3) 21 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (C)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 5 ☐

2) 14 ☐

3) 22 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (D)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 7 ☐

2) 12 ☐

3) 19 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (E)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 6 ☐

2) 13 ☐

3) 20 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (F)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 8 ☐

2) 17 ☐

3) 26 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (G)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 1 ☐

2) 9 ☐

3) 24 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (H)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 3 ☐

2) 15 ☐

3) 25 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (I)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 2 ☐

2) 11 ☐

3) 23 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) ☐

2) ☐

3) ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick: Activity Sheet 2 Solutions Template

NUMBER	3^0	3^1	3^2
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			
26			

Graph Theory

Introduction

Graph theory is the study of graphs i.e. structures which are used to model relationships between objects. These can be as straightforward as paths or roads connecting towns and cities. Or they can be used in more unconventional ways such as modelling the connection between atoms in molecules, or analysing connections between people on social networks like Facebook.

Aim of the Workshop

The aim of this workshop is to introduce students to the basic concepts of graph theory through exploring the Bridges of Königsberg problem. The students will attempt to solve the problem before being introduced to the theory behind the solution and the solution to the problem itself. The workshop will also give insights into how graph theory is applied in various disciplines.

Learning Outcomes

By the end of this workshop students should be able to:

- Represent problems in graphical form
- Define the conditions required for an Euler path to exist
- Recognise the Hand–Shaking Lemma and the 4 Colour Theorem and describe their relevance to Graph Theory
- Recognise other applications of Graph Theory.

Materials and Resources

Each student will require: paper, pens, activity sheets (activity sheet 1 preferably laminated), whiteboard marker, tissues/erasers,

Key Words

Graph
structures which are used to model relationships between objects.

Bridges of Konigsberg
A problem which is credited as the beginnings of Graph Theory derived by Leonhard Euler.

Euler/ Euler Path
Euler was a Swiss mathematician, physicist, astronomer and engineer. He presented a solution to the Bridges of Konigsberg problem in 1735 leading to the definition of an Euler Path, a path that went over each road exactly once.

Vertex/ Edges
See Appendix – Note 1.

Graph Theory: Workshop Outline

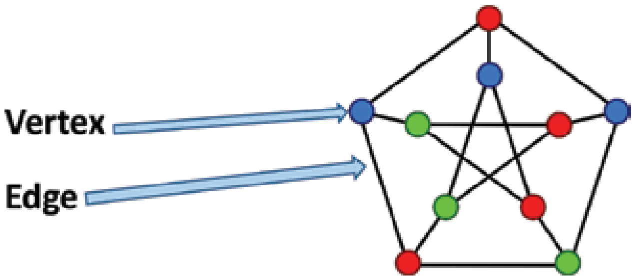
SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Introduction to workshop and Graph Theory**	– Ask students what they think graph theory might involve. Highlight the uses of graph theory in social networks such as Facebook etc. – Students should be divided into groups of 5
5 mins (00:10)	**The Bridges of Konigsberg Problem**	– Introduce the Bridges of Konigsberg problem: "People wondered if they could visit all areas of the city while only crossing each bridge exactly once" – **Activity Sheet 1:** Can you walk through the town crossing each bridge exactly once? – Students should try to trace such a path on the laminated maps (or in pencil if not laminated).
10 mins (00:20)	**Representing the problem as a Graph**	– Introduce the concepts of graphs by trying to simplify the picture of the town, with each land mass as a point and the bridges as lines connecting the pints as in the image below: – **Activity Sheet 2:** Find paths for simple shapes. Students should work together to decide if they can find a path that goes over each line once (draw on **Sheet 2A**, answers on **Sheet 2B**)
10 mins (00:30)	**Elementary Graph Theory**	– Introduce the components of a graph – **Vertices** and **edges**, **odd** and **even** degree for vertices (see **Note 1**) Degree of a vertex = number of paths entering that vertex. – **Activity Sheet 3:** Calculate the number of vertices for each of the simple graphs from before and identify the number of vertices of odd degree and even degree.

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
10 mins (00:40)	**Solution to the Bridges of Konigsberg Problem**	– Discuss the mathematician Euler and the definition of an Euler path (a path that crosses each edge exactly once) – Ask students to compare which of the earlier shapes had paths to the number of vertices of odd degree. See if students can spot the rule for deciding if there is an Euler path. – Define that an Euler path exists only if there is exactly zero or two vertices of odd degree – Note that for no odd vertices you start and finish on the same point but for two odd vertices you must start on one odd vertex and finish on the other! (see **Appendix – Note 3**) – Reveal that the Bridges of Konigsberg problem has no solution! – **Activity Sheet 4:** Which of the following (new) graphs have Euler Paths? Students should solve this using the new theory they have learned rather than tracing out the path
10 mins (00:50)	**The Hand–shaking Lemma**	– **Activity Sheet 5:** Students shake hands with everyone in their group – Students are asked to calculate the number of hands they shook and the number of handshakes that occurred in total – Students are asked to try draw this problem as a graph – **HINT:** let each person be a vertex and each handshake be an edge!
10 mins (01:00)	**The four colour theorem**	– Show the students an image of South America and pose the question: "If two bordering countries cannot be the same colour what is the fewest number of colours we need to colour in the entire continent?" – Students attempt colouring the map **(Activity Sheet 6)** – Ask students "Can we draw a graph to work out the minimum number of colours needed?" – (Note: there are only 4 colours needed)
10 mins (01:10)	**Hamilton Paths and the Icosian game**	– Task: **Activity Sheet 2** In pairs, students work together using backwards induction (starting with 2 pirates and working up to 5 pirates) to determine the optimal suggestion for the eldest pirate – Whole class discussion on strategies and solution (as noted in **Appendix – Note 4**)

Graph Theory Appendix:

Note 1: Graph theory definitions

Degree is the number of edges entering or leaving the vertex



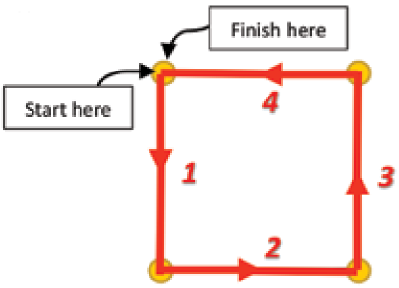
Note 2: Graph Theory Slides

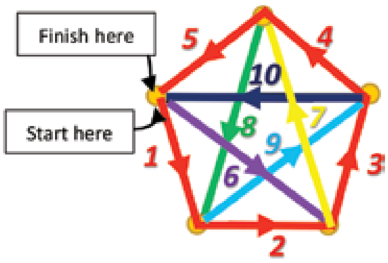
Slides for this workshop can be found at the following address:
https://prezi.com/njwia55925qs/graph–theory/?utm_campaign=share&utm_medium=copy

Please note that these slides do not include the four colour theorem or a discussion of Activity Sheet 5 or 6.

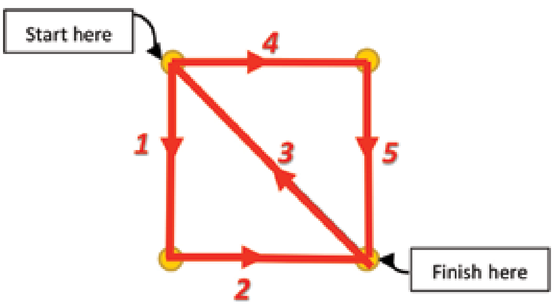
Note 3: Examples of zero odd vertices and two odd vertices

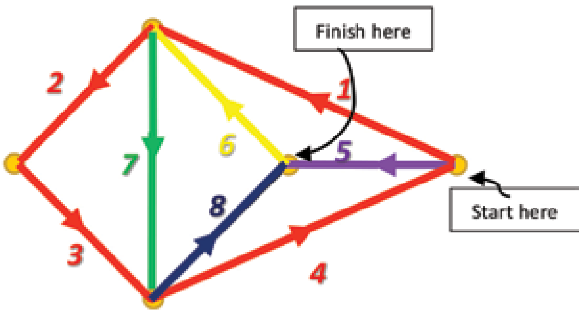
If a graph has **no** vertices of odd degree then you must *start and finish at the same vertex* to complete an Euler path



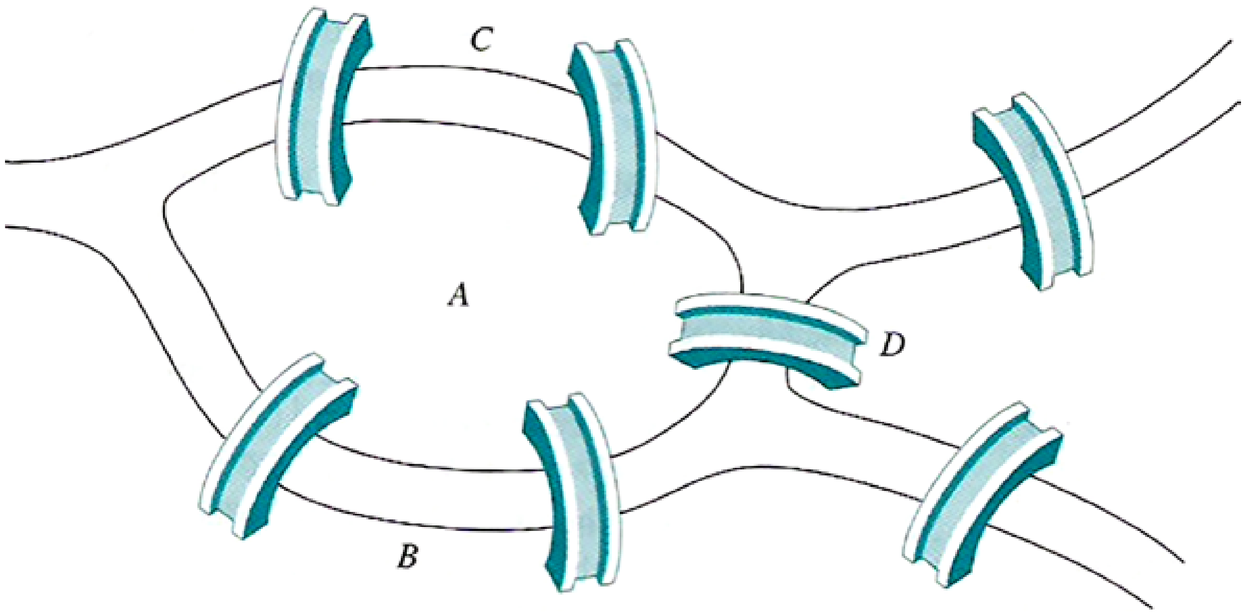


If a graph has **two** vertices of odd degree then you *must start at one of the odd vertices and finish at the other odd vertex* to complete an Euler path

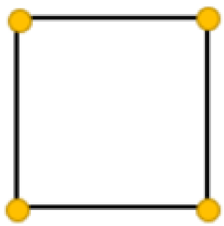




Graph Theory – Activity Sheet 1



Graph Theory – Activity Sheet 2A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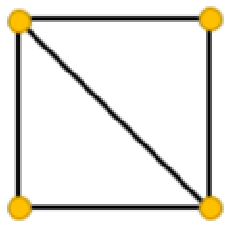
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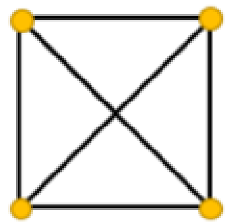
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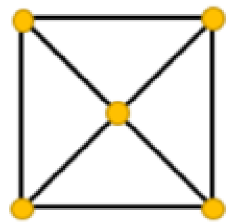
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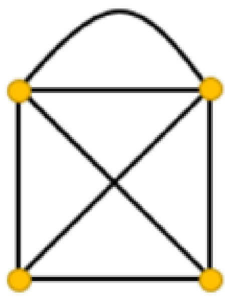
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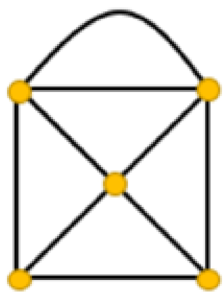
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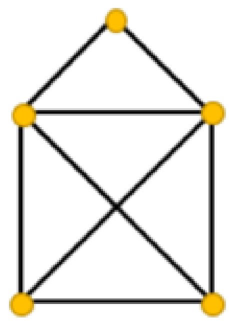
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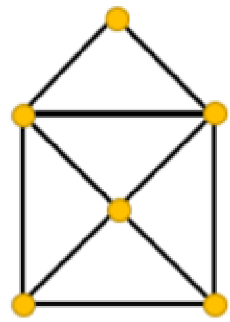
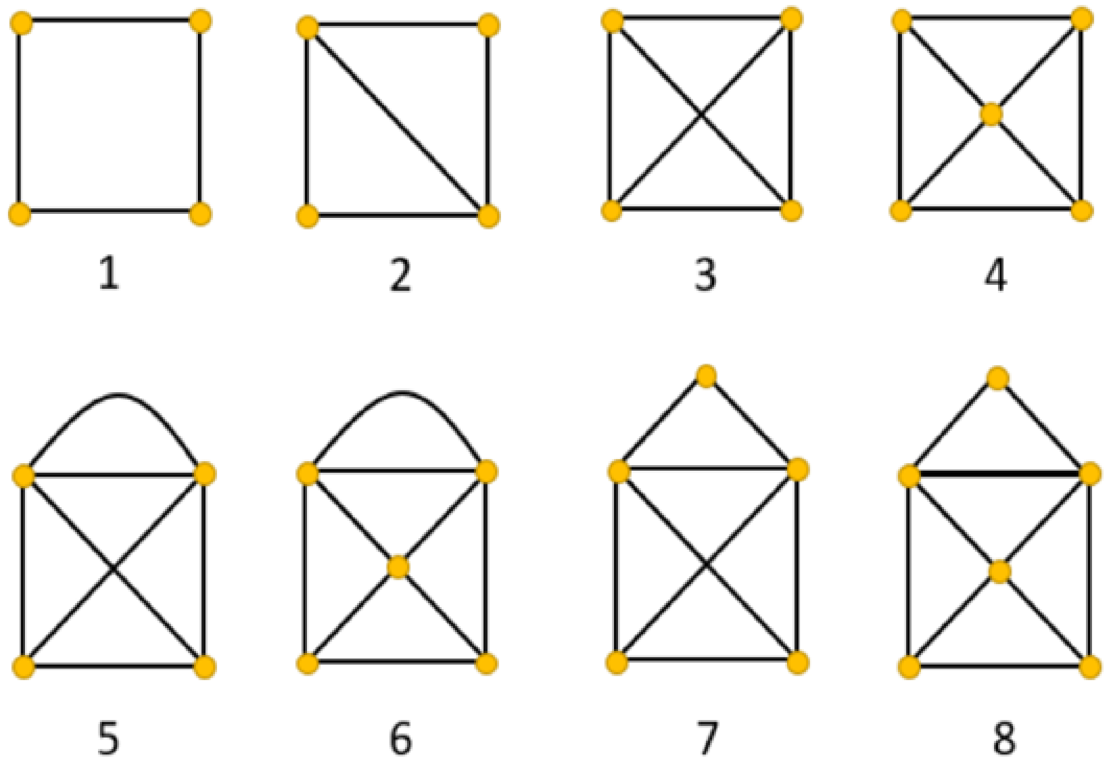

8

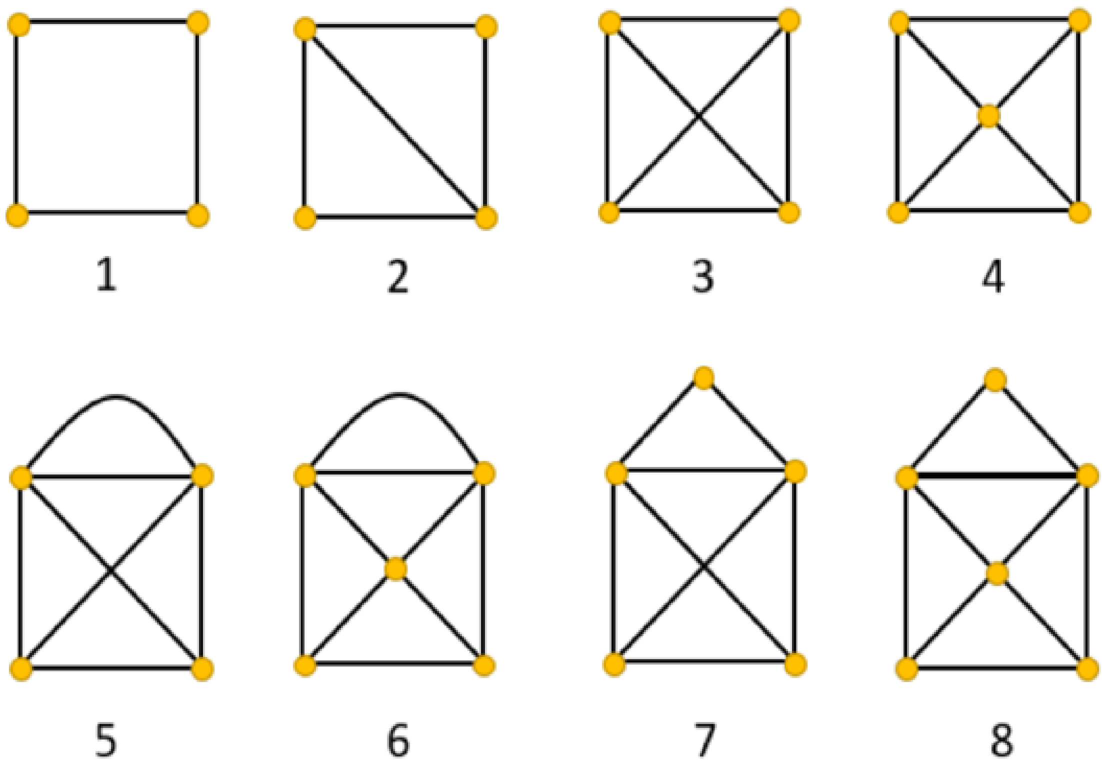
Graph Theory – Activity Sheet 2B



GRAPH	PATH; YES/NO
1	
2	
3	
4	
5	
6	
7	
8	

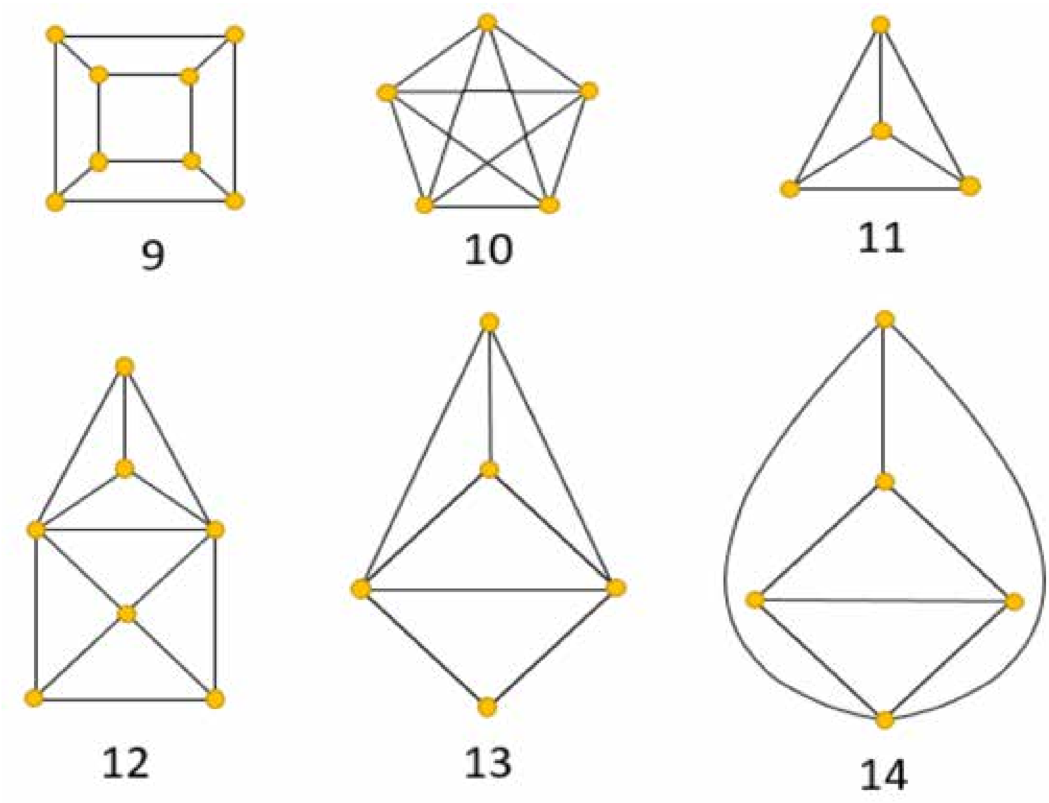
Hint – There are two which do not have a path!

Graph Theory – Activity Sheet 3



GRAPH	NUMBER OF VERTICES	NUMBER WITH EVEN DEGREE	NUMBER WITH ODD DEGREE	PATH; YES/NO
1				
2				
3				
4				
5				
6				
7				
8				

Graph Theory – Activity Sheet 4



GRAPH	NUMBER OF VERTICES	NUMBER WITH EVEN DEGREE	NUMBER WITH ODD DEGREE	PATH; YES/NO
9				
10				
11				
12				
13				
14				

Graph Theory – Activity Sheet 5

"Everyone shake hands with each person in their group..."

How many hands did you shake?

How many handshakes in total?

Can you draw a graph to represent the handshakes?

Graph Theory – Activity Sheet 6

What is the minimum number of colours needed to colour the map below, so that neighbouring countries do not have the same colour?

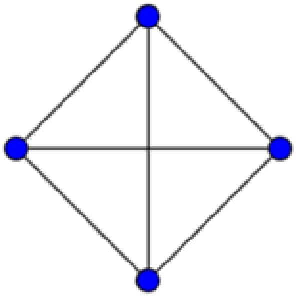
Credit: http://www.printablemaps.net/south–america–maps/

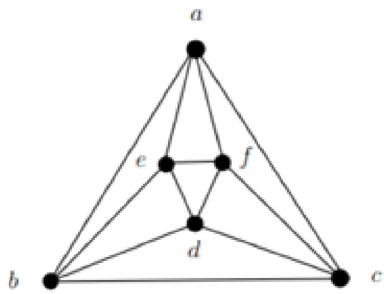
Can you draw a graph to work out the answer to the above question?

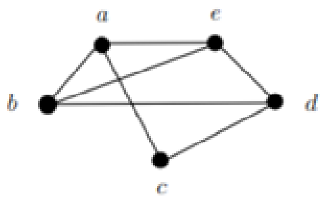
HINT: think about what a vertex or edge might represent on your graph

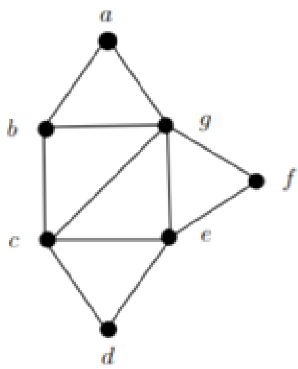
Graph Theory – Activity Sheet 7

1. Decide whether the graphs below have Hamiltonian paths or Hamiltonian cycles or none of them.

a) 

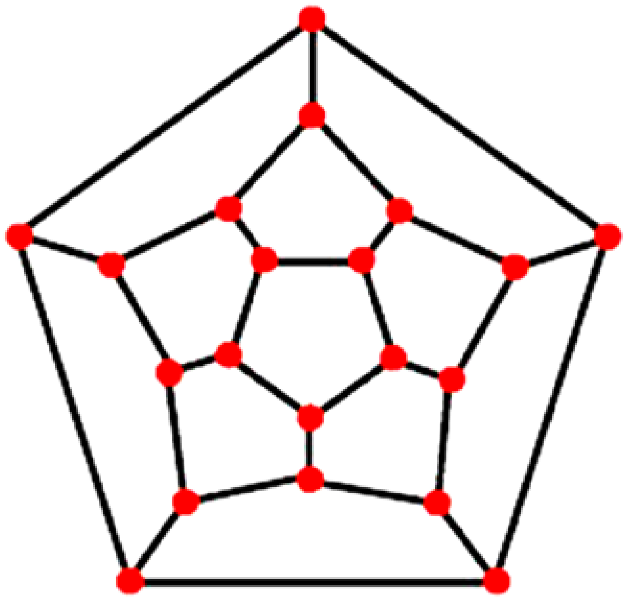
b) 

c) 

d) 

HINT: A Hamilton Cycle is a Hamilton path that begins and ends at the same vertex

2. Try to find a Hamiltonian cycle in Hamilton's famous Icosian game.



Geometric Series and Infinite Games

Introduction

One of the most interesting questions often asked is "what is infinity?" This question has not only perplexed learners, but also mathematicians and scientists who are still trying to make sense of this fascinating concept. The number of grains of sand on a beach, the amount of atoms in our bodies or even the sum of all the stars in the observable universe are finite. In fact, everything that we see around us appears to be finite. The question is therefore, does infinity really exist? Well, mathematics has given us a better understanding of the concept of infinity and its rather peculiar properties.

Aim of the Workshop

The aim of this workshop is to introduce students to the concept of infinity and its applications in counting problems and infinite sums. In particular, the workshop will outline some of the unusual behaviours of infinity as seen in the Menger Sponge and the Hilbert's Hotel paradoxes. Different types of infinite sums will also be discussed, including sums which converge to a finite number, sums which diverge to infinity, and sums that do neither (e.g. Grandi's Series).

Learning Outcomes

By the end of this workshop students should be able to:

- Recognise the Sigma notation and understand how it is used
- Explain, in their own words, what is meant by infinity
- Describe some of the unusual properties of infinity (Hilbert Hotel, Menger Sponge, etc.)

Materials and Resources

Optional: Hilbert's hotel video clip
https://www.youtube.com/watch?v=faQBrAQ87l4

Keywords

Infinity
An abstract concept used to describe something that is unbounded or greater than any known quantity.

Geometric Sequence
An ordered list of numbers in which each term is found by multiplying the previous term by a constant.

Geometric Series
A geometric series is the sum of the numbers in a geometric sequence.

Geometric Series & Infinite Games: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Introduction to the Concept of Infinity**	– Define what is meant by Infinity (see keywords) – Introduce students to the Menger Sponge (**see Appendix– Note 1**)
15 mins (00:20)	**Geometric Series**	– Ask students "What is a sequence?" – Once student feedback is discussed, explain that a sequence is an ordered list of numbers and that the sum of such sequences is called a 'series' – Define what is meant by both a Geometric Sequence and Geometric Series (see keywords)
15–20 mins (00:40)	**Sigma Notation**	– Introduce students to the Sigma Notation Σ – Explain to the students that sigma means "sum of" and is used to find the sum of a sequence – **Activity Sheet 1:** Divide students into pairs and ask them to complete activity sheet 1 (see **Appendix– Note 2**)
5 mins (00:45)	**Discussion on Activity 1**	– Once Activity 1 is complete, ask students: • "If we *decrease* the number on top of Sigma, will our result become larger or smaller than before?" • "What if we *increase* the number on below Sigma, will our result become larger or smaller than before?" • (it is important to note the different types of sequences) • *"What about Infinity?"*
15 mins (01:00)	**Hilbert's Hotel**	– Introduce students to the Hilbert's Hotel paradox (see **Appendix – Note 3**) – Divide students into groups and ask them "If Hilbert's Hotel is fully booked, is it possible to accommodate another guest? (see **Appendix– Note 4**) – Ask students "A bus arrives containing an infinite number of guests who wish to stay at the hotel. How can we accommodate them if all rooms are fully booked?" (see **Appendix – Note 5**) – Whole class discussion on possible solutions. – Optional: Play a brief video clip on Hilbert's Hotel which explains the solutions to the above situations (link above in additional resources)

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
10 mins (01:10)	**Achilles and the Tortoise (Zeno's Paradox)**	– **Activity Sheet 2** In pairs, students complete Activity Sheet 2 – Whole class discussion on the activity – Explain the idea behind Zeno's paradox (**see Appendix– Note 6**)
10 mins (01:20)	**Grandi's Series**	– Explain the Grandi's Series to the students 1 – 1 + 1 – 1 + 1 – 1 + 1 – 1 + ... (see **Appendix– Note 7**) – **Activity sheet 3 –** In pairs, students complete Activity Sheet 3

Geometric Series and Infinite Games Workshop Appendix

Note 1: Menger Sponge

Menger Sponge is a theoretical shape that has infinite surface area and no volume. It is also known as a fractal curve meaning it exhibits the same repeating pattern at every scale.

The Menger sponge begins with a solid cube. This is divided into 27 smaller cubes and the centre from each of these cube faces is removed.

This process is repeated with the remaining cubes, leading to similar shape to that shown in figure 1. In doing so, the volume reduces and surface area increases each time. Given that this process can extend to infinity, Menger Sponge will thus have infinite surface area and zero volume.

Figure 1: Diagram of the Menger Sponge

Note 2: Activity Sheet 1 Question 3 Solutions

(a). $\sum_{n=1}^{\infty} 2^n$

(b). $\sum_{n=0}^{\infty} (5)3^n$

(c). $\sum_{n=0}^{\infty} \frac{81}{3^n}$

(d). $\sum_{n=0}^{\infty} (-1)^n$

Note 3: An Introduction to Hilbert's Hotel

Hilbert's Hotel is a mathematical paradox that is used to demonstrate some of the unusual properties of infinity. In a standard hotel, there are a finite number of rooms. Once each of the rooms has been assigned to a guest, the hotel is considered fully booked. However, in the case of Hilbert's hotel, there are an infinite number of rooms.

Note 4: Accommodating an Extra Guest in Hilbert's Hotel

If all the rooms in Hilbert's Hotel are booked out, it might appear that no more guests can be accommodated. Fortunately, however, a room can be provided for an additional guest by moving the guest staying in room 1 to room 2. The guest in room 2 then moves to room 3 and so on. Room 1 will thus be available for the new guest once everyone else has moved accordingly. Representing this mathematically, the guest in room **n** will be moved to room **n+1**. This demonstrates how it is possible to accommodate a new guest even if the hotel is already fully booked, something that could not happen in a hotel with a finite number of rooms.

Note 5: Accommodating an Infinite Number of Guests in Hilbert's Hotel

If a bus then arrives with an infinite number of guests, it is still possible to accommodate them in Hilbert's Hotel despite each room being occupied. This time, instead of moving each guest to the room beside them (i.e. n+1), we ask them to move to the room which is double their current one. In other words, the guest in room 2 moves to room 4, the guest in room 3 moves to room 6 and so on. This leaves the infinitely many odd numbered rooms free which can thus accommodate the infinite number of guests. Representing this mathematically, the guest in room **n** will be moved to room **2n**. This paradox is an interesting way to demonstrate the unusual properties of infinity.

Note 6: Zeno's Paradox

Greek philosopher Zeno designed a paradox to describe a way in which a tortoise could win a 1 Kilometre race against the legendary hero "Achilles".

The tortoise is given a head start of 500 metres. Once the race begins, Achilles would first have to first cover the distance to the point that the tortoise started. Meanwhile, the tortoise would have moved a little further from the 500 metre mark. Achilles would then have to cover that distance too, giving the tortoise time to move forward even more.

Whilst the gap between the two may reduce in size over time, Zeno pointed out that this process could go on infinitely long, given that the tortoise would be able to move forward each time Achilles is catching up. Thus Achilles could never win. This paradox led to the realisation that something finite could be divided an infinite number of times. (Where "…." means the pattern is recurring.)

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}.........$$

Note 7: Grandi’s Series

The Grandi’s Series, shown below, is an infinite series named after Italian mathematician Guido Grandis.

$$1-1+1-1+1-1+1-1+1-1........$$

By using parentheses, there are different ways of adding the Grandi’s series, each of which produces a contradictory result. Hence it is also known as a divergent series, meaning it does not have a sum.

For example: $(1 - 1) + (1 - 1) + (1 - 1) + ... = 0 + 0 + 0 + ... = 0$.

On the other hand, the following parentheses give:

$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + ... = 1 + 0 + 0 + 0 + ... = 1$.

Thus, we can obtain both 0 and 1.

Sources and Additional Resources

http://wordplay.blogs.nytimes.com/2016/05/30/frenkel–cantor/

https://www.youtube.com/watch?v=faQBrAQ87l4 (Hilbert’s hotel video clip)

http://mathandmultimedia.com/2014/05/26/grand–hotel–paradox/

http://www.mathsisfun.com/algebra/sigma–notation.html

http://platonicrealms.com/encyclopedia/zenos–paradox–of–the–tortoise–and–achilles

Geometric Series & Infinite Games: Activity Sheet 1

Geometric Series

1. Can you find the next 2 terms in each of the following sequences?

(a) 2, 4, 8, 16, ___, ___

(b) 5, 15, 45, 135, ___, ___

(c) 81, 27, 9, 3, ___, ___

(d) 1, –1, 1, –1, 1, ___, ___

2. For each of the following, write down what you think the geometric sequence is (and solve it if you can!)

$$\sum_{n=1}^{4} 3n$$ (a) =

$$\sum_{n=1}^{3} (2n+1)$$ (b) =

$$\sum_{n=2}^{5} n^2$$ (c) =

3. Can you write the sequences in Question 1 using sigma notation?

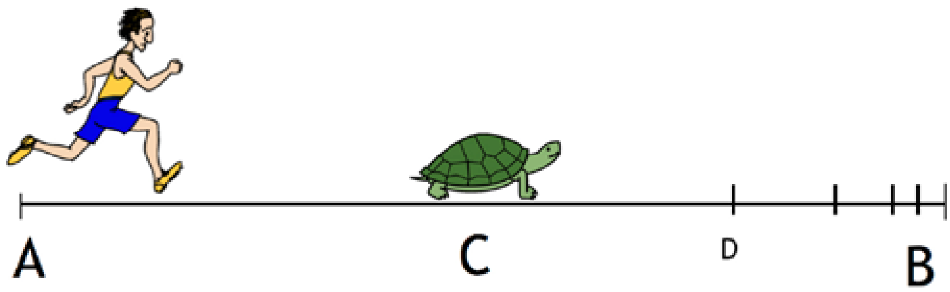
Geometric Series & Infinite Games: Activity Sheet 2

Achilles and the Tortoise

Achilles and his tortoise Hillary decide to have a race over 1 Kilometre. Given that Achilles runs twice as fast as Hillary, he decides to give her a head start of 500 metres from point C (the halfway mark).

The race begins and after Achilles arrives at C, Hillary has moved a certain distance ahead to a point D (midpoint between C and B).

Similarly, Achilles must then travel to point D. By this time, Hillary has moved onto point E (midpoint between D and B). Label this point.



1. **What is the distance from C to D? (In km)**

2. **What is the distance from D to E in? (In km)**

3. **Continuing in a similar pattern, how many "halfway points" will Achilles have to pass in order to reach B?**

Geometric Series & Infinite Games: Activity Sheet 2

4. Based on your previous answer, do you think Achilles will ever reach Hillary? Explain your reasoning.

5. What would happen if we were to write this as a geometric series?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots = ???$$

6. Looking back at Activity 1, how would we write this with Sigma notation?

$$\sum_{n=}$$

Geometric Series & Infinite Games: Activity Sheet 3

Grandi's Series

Below is the famous Grandi's series.

1-1+1-1+1-1+1-1+1-1........

1. Can you figure out the following "partial sums" of the Grandi's Series?

(a) $\sum_{n=0}^{2} (-1)^n$ =

(d) $\sum_{n=0}^{10} (-1)^n$ =

(b) $\sum_{n=0}^{3} (-1)^n$ =

(e) $\sum_{n=0}^{15} (-1)^n$ =

(c) $\sum_{n=0}^{5} (-1)^n$ =

(f) $\sum_{n=0}^{38} (-1)^n$ =

2. Can you see a pattern? Discuss with your partner.

(g) $\sum_{n=0}^{\infty} (-1)^n$ =

Probability Theory

Introduction

Probability theory is an area of mathematics that deals with random events where we calculate how likely it is for an event to occur. It is used widely in the world of statistics because it can be used to make predictions or analyse the most likely events to occur in an experiment and compare these results to theoretical predictions. Probability Theory can be traced back to peoples' attempts to analyse games of chance and dice (before the mathematics of probability was invented).

Aim of Workshop:

The aim of this workshop is to introduce students to the basics of Probability Theory and its applications through the Monty Hall Problem. The Monty Hall problem is a famous problem based on the American television show "Let's Make a Deal" and is named after the show's host – Monty Hall.

Learning Outcomes

By the end of this workshop students should be able to:

- Discuss, in their own words, the concept of probability and sample spaces
- Describe the Monty Hall problem
- Recognise the applications of Probability Theory.

Keywords

Sample space

The list of all possible outcomes from some experiment E.g. the sample space for rolling a six sided dice is given as: $S = \{1, 2, 3, 4, 5, 6\}$.

Probability Theory: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Play the Monty Hall Game**	– Ask for 3 students to volunteer to play "Let's Make a Deal" – Ask each student to give the best definition of probability that they can – Allow the rest of the class to decide which student gave the best definition – The student who gets the most votes will play the game (See **Appendix – Note 1** for the game show rules)
10 mins (00:15)	**Probability Theory definitions**	– Introduce the topic of the workshop and give some definitions required for the lesson, such as sample space, event space etc. (See **Appendix – Note 2**) – Students should attempt **Activity Sheet 1**
10 mins (00:25)	**Play the 3 card version of the Monty Hall game**	– Allow the students to play the game in pairs using 3 cards. – Ask: "Do we think you win more often if you switch or stay when offered the chance? Or will it make any difference at all?" – Students should try both strategies and experiment as to how often they win in e.g. 10 plays when they stay every time and then when you switch every time.
10 mins (00:35)	**Solve the Monty Hall 3 card problem**	– Ask students to attempt **Activity Sheet 2** (Remind students to keep in mind the basic concepts that they have seen before)
10 mins (00:45)	**Explain the solution**	– Discuss the solutions to the Monty Hall problem and why discuss switching is the best thing to do over many times playing the game (see **Appendix – Note 3**)
10 mins (00:55)	**The 4 card Monty Hall problem**	– Let the student play the 4 card version of the game to reinforce the idea that swapping is the better option. – Students should attempt **Activity Sheet 3**
15 mins (01:10)	**N door Monty Hall problem**	– Ask students to attempt **Activity Sheet 4** and see if the can work out what happens to the chances of winning when you switch for a large amount of doors. – Discuss some of the applications of probability (see **Appendix – Note 4**)

Probability Theory Appendix

Note 1 – Let's Make a Deal/ Monty Hall problem Rules

- There are three doors and behind one of the doors is a brand new car (or other desirable item) and behind the other two doors are goats (or other nonsense prize)
- The contestant is given the choice to choose one of the three doors.
- After the contestant chooses a door, the host will then reveal what's behind one of the other two remaining doors, but he will only ever reveal a goat and never the car (the host always knows where the car is).

  For example, if the car is behind door 2 and the contestant chose door 1, then the host will open door 3 (revealing one of the goats).

  If the contestant initially chose (door 2) that contains the car, then the host will open either of the other two doors (door 1 or door 3) revealing one of the goats
- After the host reveals a goat, the host then gives the contestant the option of staying with their original door or switching to the one remaining door.
- The big question is: Is it in our best interest to switch or stay?

Note 2 – Probability Theory Definitions

The main definition we need for working out probabilities is also given at the top of Activity Sheet 1:

$$\text{Probability of an Event} = \frac{\text{Total number of successful outcomes}}{\text{Total number of possible outcomes}}$$

For example, the probability of rolling an even number on a die is worked out by finding the individual components of the definition:

Total number of successful outcomes = {2, 4, 6} = 3 possible outcomes

Total number of possible outcomes = {1, 2, 3, 4, 5, 6} = 6 total outcomes

Therefore:

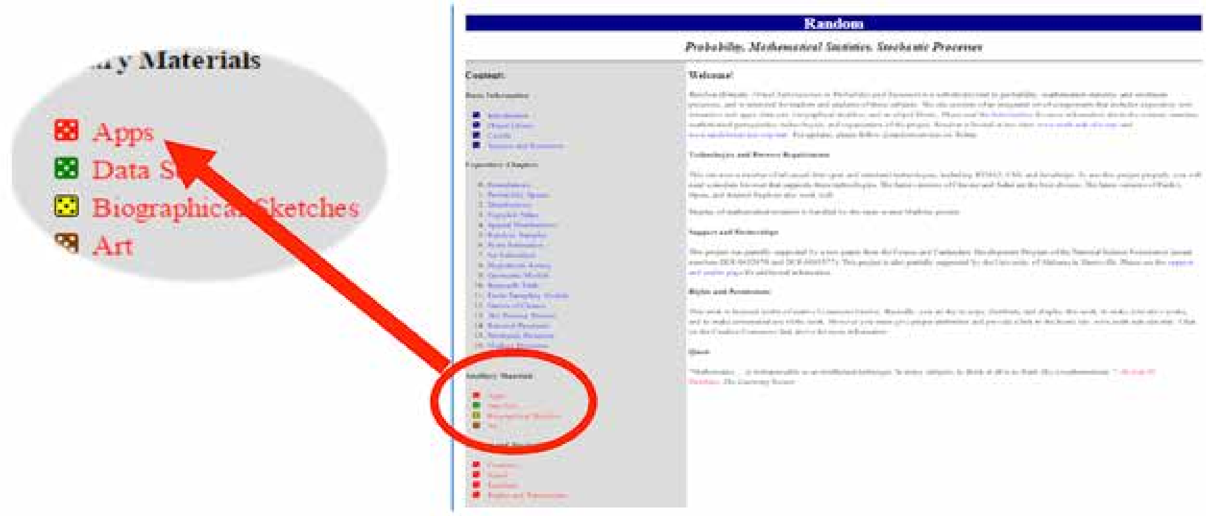
$$\text{Probability of rolling an even number} = \frac{3}{6} = \frac{1}{2} = 0.5$$

Additionally, we can make use of the rule that when we run an experiment the sum of the probabilities of each outcome occurring must add up to 1.

In the above example the probability of rolling an even number was ½ and similarly the probability of the other outcome: rolling an odd number is also ½, and the sum of these probabilities is 1.

Note 3 – Monty Hall Problem

The following is a website that provides a number of statistics apps and simulations:
http://www.math.uah.edu/stat/



Upon opening the website, choose the applets section on the left–hand side of the screen:

The list of apps that opens come from a large number of different areas of statistics but the one we are interested is under the section ***"Games of Chance" / "Monty Hall Game" or "Monty Hall Experiment"***

The **"Monty Hall Game"** is a way to play the Monty Hall Game online. The applet comes with a full description of the game and instructions on how to use the applet. This app will allow you to run individual games of Let's Make a Deal and will work well if you do not have the ability to run the game physically in the classroom.

The **"Monty Hall Experiment"** will allow you to run the game many times with the option of setting that the player always stays or always swaps doors. (Please see the applet description to get more details on the various labels on screen.)

There are also resources available on:
https://www.youtube.com/watch?annotation_id=annotation_3286749133&feature=iv&src_vid=4Lb–6rxZxx0&v=ugbWqWCcxrg

Note 4: Applications of Probability Theory

Weather Prediction:

- What is the probability that it will rain tomorrow?
- What are the chances that an earthquake will cause a tsunami?

Sport:

- How likely is each team in the next World Cup to win?
- How do bookies calculate odds on various sports games?

Entertainment:

- How likely is the next Leonardo di Caprio movie to win an Oscar?
- How likely is it for an Irish language movie to win at the Cannes film festival?
- What is the probability that U2 will play in Croke Park next year?

Note 5: Activity Sheet 4 Solutions

(N–Door) Monty Hall Problem

Problem

In this case there are initially N doors to pick from and the host opens N–2 doors (one chosen by the contestant). Everything else in the problem remains unchanged. N can be any number – in this case how many doors there are initially, for example N=12.

Solution

Answer: The contestant should switch because P (win if stay) = $\boxed{\frac{1}{N}}$

& P (win if switch) = $\boxed{\frac{N-1}{N}}$

Probability Theory – Activity Sheet 1

1 What are all the outcomes of throwing an 8–sided die? This is known as a sample space.

2. What is the probability of landing on an even number on an 8–sided die?

3 What is the probability of rolling a 5 or greater on a 4–sided die?

4. In a certain game it counts as a win if a contestant rolls higher than an 8 on a 12–sided die. What is the probability of winning the game?

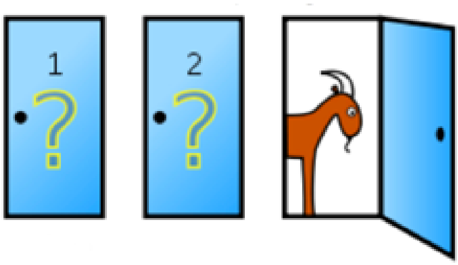
5. What is the probability of the sun rising tomorrow morning?

Probability Theory – Activity Sheet 2

(3–Door) Monty Hall Problem

Problem

- You're given the choice of three doors: behind one door is a car; behind the other two (losing) doors are goats.
- You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. The host always picks a (losing) door with no prize behind it and always knows where the car is.
- The host will always offer the contestant a chance to switch after opening one of the two remaining doors.



Q. Are you more likely to win by switching?

Solution to Monty Hall 3–card Problem

Q. What is the sample space when you switch? When you stay?

Let's play the game with three cards: one black (winning) card and two red (losing) cards. Let Card 1 be the winning card.

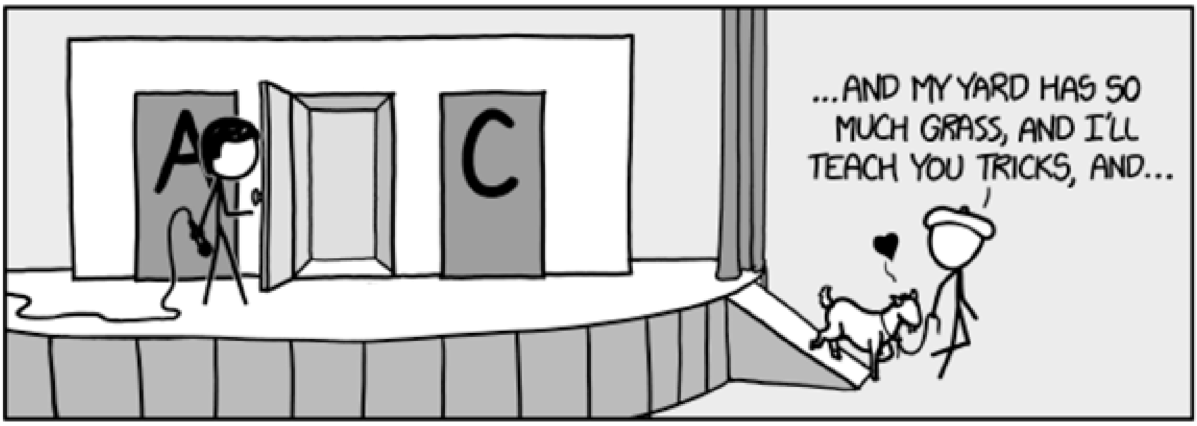
For example, if a contestant chooses card 1 and then decides to not switch when offered, then they will win the car. Enter "Win" or "Lose" for each option in the below table:

CONTESTANT'S INITIAL CHOICE OF CARD	STAY	SWITCH
Card 1		
Card 1		
Card 2		
Card 2		
Card 3		
Card 3		
P(Win)=		

The solution will be similar if card 2 or card 3 is the winning card. The probability of winning, when switching/staying remains the same regardless of which card is the winning card.
Now try entering the outcomes in the condensed table below:

CONTESTANT'S **INITIAL** CHOICE OF CARD	STAY	SWITCH
Card 1		
Card 2		
Card 3		
P(Win)=		

Though nobody said you didn't want to win the goat...



Credit: http://xkcd.com/1282/

http://imgs.xkcd.com/comics/monty_hall.png

Probability Theory – Activity Sheet 3

(4–Card) Monty Hall Problem

Problem

In the 4–card Monty Hall problem, you're given the choice of **four cards**. One card is black (the winning card); the other 3 cards are red (losing cards). You pick a card, say No. 1, and the host, who knows what the cards are, **picks two other cards**, say No. 3 and No. 4, which are both red (losing cards). He then says to you, "Do you want to pick card No. 2?"

Q. Is it to your advantage to switch your choice?

Fill in the following table just like in the last activity sheet. Just keep in mind that now you have 4 cards rather than 3 as before.

Please note the winning card is card one.

CONTESTANT'S INITIAL CHOICE OF CARD	STAY	SWITCH
1		
2		
3		
4		
P(Win)=		

Solution

Answer: The contestant should switch because P (win if stay) = ☐

and P (win if switch) = ☐

Probability Theory – Activity Sheet 4

(N–Door) Monty Hall Problem

Problem

In this case there are initially N doors to pick from and everything else in the problem remains unchanged. Can you come up with a general formula to help the contestant decide if they should switch or stay?

Q. Is it to the advantage of the contestant to switch their choice?

P (win if stay) =

and P (win if switch) =

The contestant should always

because

Hint:

As we know the probabilities of winning the 3–Door, and 4–Door, try putting in N=3, and N=4 into the N–Door solution, and seeing if they match the previous probabilities.

Liar's Dice and Binomial Random Variables

Introduction

Probability is one of the most important areas of study in Mathematics as it governs almost every decision that is made in the world today. From weather prediction, to predicting stock prices and winners of sports competitions, probability comes up in every area where prediction of any kind is involved. Therefore, it is imperative that students begin to see many of the ways that we can express probabilities and group events into certain types or "distributions". One such distribution is the Binomial distribution.

Aim of the Workshop

The aim of this workshop is to introduce binomial random variables in the context of a game of chance. Liar's Dice (or Perudo) is an ancient game where having an understanding about the probability of certain combinations of dice rolls is the difference between winning and losing. We introduce the binomial formula and demonstrate how we can use it to work out probabilities that would be otherwise complicated and cumbersome to calculate.

Learning Outcomes

By the end of this workshop students should be able to:

- Explain, in their own words, what each term of the binomial formula represents
- Be able to apply this formula to examples involving the rolling of dice
- Describe the application of the theory to the game Liar's Dice.

Materials and Resources

Each student will require: at least 4 six–sided dice (Max. 6), a cup or other non–transparent container (or they can use their hands to cover their dice), activity sheets, paper, pens

Key Words

Distribution
describes the shape of the data when plotted on a histogram.

Bernoulli trial
Experiment where there are only two possible outcomes, usually with one defined as "success" and one defined as "failure" and where the probability of success is the same every time we perform the experiment.

Binomial Random Variable
is the number of successes (usually labelled k) in n independent Bernoulli trials.

Independent events
the occurrence of one event does not affect the probability of any of the others.

Binomial forumula

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Liar's Dice and Binomial Random Variables: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
10 mins (00:10)	**Introduction of Bernoulli trials and Binomial Formula**	– Introduction to what a Bernoulli trial is and how we recognise it: Two outcomes: success or failure Examples: win or lose a game etc. – Extend the idea of one Bernoulli trial to many **independent trials** and introduce the binomial formula. (Remind the students of the meaning of **independent events** in probability with examples). – Introduce the Binomial formula and explain what each part represents (this may also be a useful revision exercise). – Discuss the other uses for the formula in fields such as medicine, weather and sports.
15 mins (00:25)	**Activity Sheet 1**	– Students attempt the questions on the **Activity Sheet**. – (They may need guidance in interpreting the various parts of the formula and filling in the correct values.)
10 mins (00:35)	**Explaining the rules of Liar's Dice**	– Discuss the rule sheet for the game (See **Appendix – Note 1**) and clarify these by going through some examples to demonstrate calling a bluff or calling a spot–on bet work in the game (See **Appendix – Note 2**). (You may wish to use one of the examples in the activity sheet to emphasise how likely/unlikely certain bids are to have occurred.)
25 mins (01:00)	**Let the students play the game**	– Students play the game. – After playing for 5–10 minutes, pause the games and ask students to try **Activity Sheet 3**. – This newly found knowledge from Activity Sheet 3 should provide students with opportunity to better strategize on bets or bluffs during the next game (**see Note 3**).

Liar’s Dice Appendix

Note 1: Liars Dice – How to play

- Each player starts with between 4 and 6 dice.
- To decide who goes first:

  Each player rolls one dice and the player that rolls highest will start the game. In the event of a tie, re–roll the dice until a clear winner is found.
- At the start of each round all players roll all of their dice inside of their dice cups and place them covered on the table.
- After taking time to look at their dice one player starts by placing a bet of a certain number of dice of a chosen face value (1, 2, 3, 4, 5, 6) being on the table (i.e. under ALL the cups)
- Each player then takes turns increasing the bet until one player believes the person before them has placed and incorrect bet (a **bluff**) or has predicted an exact number of dice on the table (a **spot on** bet) (see the **rules** section for more details).
- Once one player calls out the previous player’s bet, every player reveals their dice and the correct number of dice matching the current bet are counted. Depending on the call made and the number of dice matching the bet of the bluff or spot on bet, one player (or more) will lose a dice depending on the result (see **rules** below).
- After the result of the call is displayed by all players, all players roll their dice again and the next round begins with the player to the left of the player that started the previous round (counter–clockwise).
- The game continues until only one player has dice remaining and is declared the winner.

Placing bets
On their turn the player can do one of the following:
- Bet on the same number of dice of a HIGHER face value than the previous bet **(a player can never lower the face value only)**
- Bet on a higher number of dice of the same face value than the previous bet
- Bet on a higher number of dice AND any face value (higher OR lower)

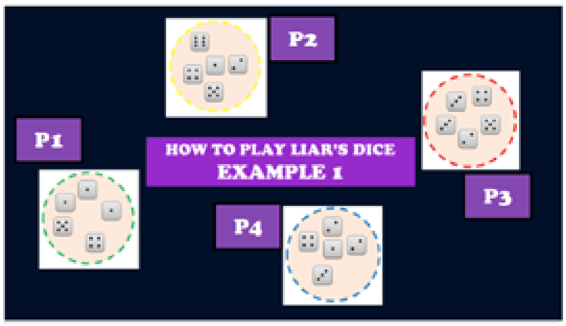
Calling on the previous bet
Instead of placing a bet a player can choose to challenge the previous bet as either a bluff (usually with the call of “bluff”) or claim the previous bet was “spot on” (i.e. there is exactly the number of dice they bet on the table). The outcomes are as follows:

Call of “liar”
- If there are fewer than the bet number of dice on the table, then the player who placed the bet loses a dice
- If there are equal or more dice than the placed bet on the table, then the player who called “bluff” loses a dice

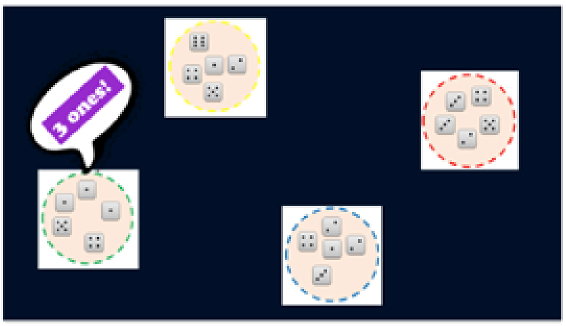
Call of “spot on” bet
- If the bet matches the number of dice on the table, then every player other than the player who called “spot on” loses a dice
- If the bet does not match the number of dice on the table, then the player who called “spot on” loses a dice.

Note 2: Liar's Dice – How to play Examples

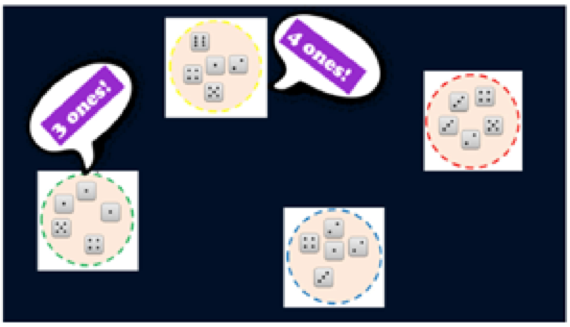
Example 1: The following is an example of how calling bluff on a bet works:



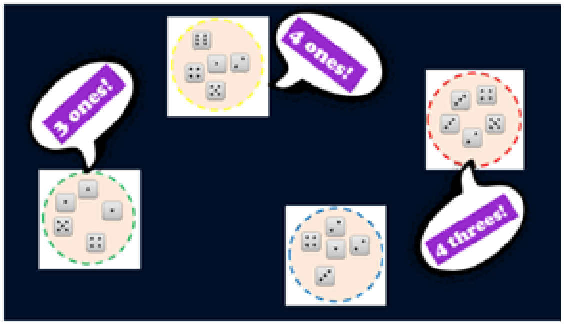
All students roll their dice and hide them under their cups. Each student checks their dice and one takes their turn placing a bet.



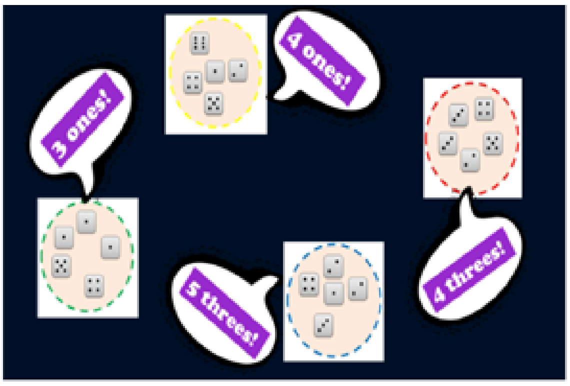
Player 1 rolled three ones, so they are certain that there are at least 3 ones on the table so they place their bet.



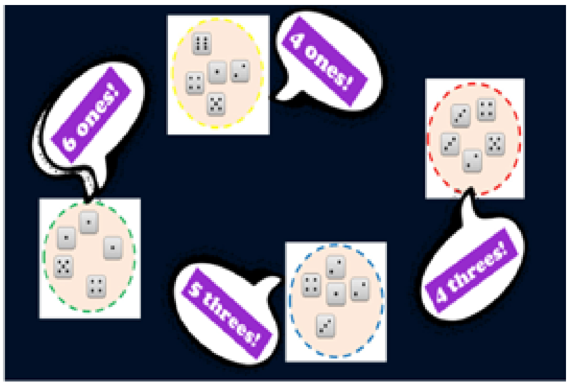
Player 2 increases the bet of the number of ones on the table from 3 to 4.



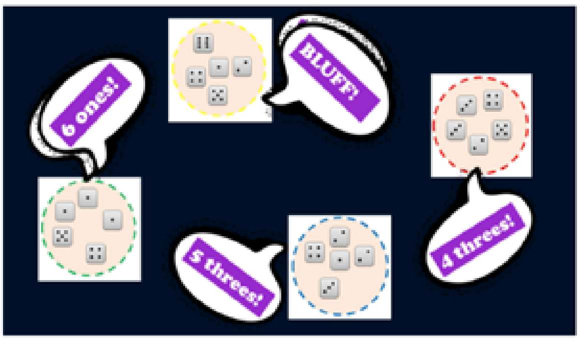
Player 3 decides to increase the face value of the dice from one to three since they have rolled 2 threes but 0 ones.



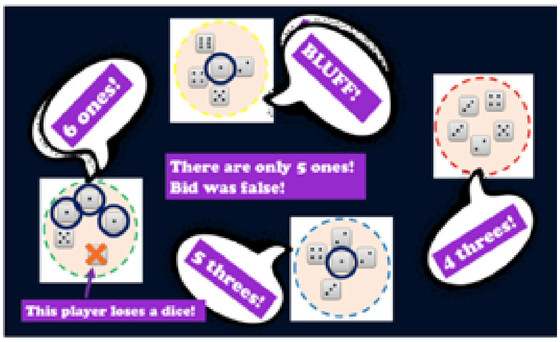
Player 4 increases the bid to 5 threes. Despite only rolling 1 three, they are hoping that there are at least 4 more on the table.



Player 1 increases the bid to 6 ones. (Note that they could not say 5 ones as you cannot decrease the face value only.)

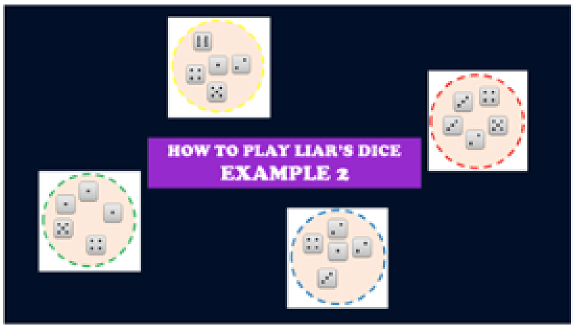


Player 2 calls bluff, believing that the previous bid of “6 ones” is false. (They think there are less than 6 ones on the table.)

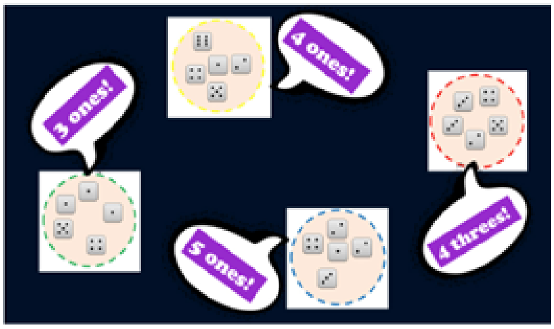


All players reveal their dice and the number of ones is counted. There are 5 ones so the bid was a bluff, so player 1 loses a dice! This dice is removed from the game and cannot be used again.

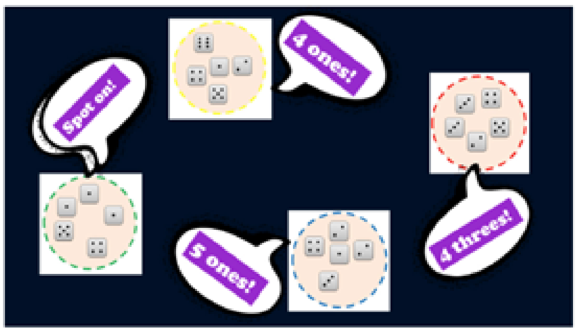
Example 2: The following is an example of calling a spot on bid



All students roll their dice and hide them under their cups. Each student checks their dice and one takes their turn placing a bet.



All bids progress as in example 1, however player 4 now bids that there are 5 ones on the table.



Player 1 calls the previous bid spot on, claiming that there are exactly 5 ones on the table

Since there are 5 ones on the table the spot on bid was correct. This means that every other player loses a dice. These dice are removed from the game and cannot be used again.

Note 3: Answers to Activity 3 – what the teacher sees

VARIABLE	SYMBOL	VALUE FOR TEACHER'S PROBLEM
Number of trials	n	**15**
Number of successes wanted	k	**5**
Probability of Success on one trial	p	**1/6**
Probability of Failure on one trial	$q = 1 - p$	**5/6**

P (Paul is correct) = *P (* **5** *ones in* **15** *dice)* =

$$\binom{15}{5}\left(\frac{1}{6}\right)^{5}\left(\frac{5}{6}\right)^{15-5} = \ldots\ [\textit{calculator}]\ \ldots \approx 0.06$$

The Binomial Formula – Activity 1

In all of the following examples we make use of the Binomial Random Variable formula for n **independent** random trials

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Where

X = event of interest (e.g. getting heads on a coin toss)
n = total number of trials or experiments
k = number of desired successes { k = 0, 1, 2, ..., n}
p = probability of success
1 – p = probability of "failure"

A fair coin is thrown 10 times.
We count the number of heads we get in the 10 coin tosses. Can you find the following probabilities?

(a) the probability that you get heads 4 times

n =
k =
p =
1 – p =

Hence using the formula:

$$P(\#heads = 4) =$$

Hint: In probability we can use the following rule for independent Binomial experiments. If we have 7 trials, say, then:

$$P(X \geq 5) = P(X = 5) + P(X = 6) + P(X = 7)$$

and this is true for any number of events

The Binomial Formula – Activity 1

(b) the probability of 8 or more heads

n =
k =
p =
1 – p =
Hence using the formula:

$$P(\#heads \geq 8\ ) =$$

Hint: In probability we can use the following rule for independent Binomial experiments. If we have 7 trials, say, then:

$$P(X \geq 5) = P(X = 5) + P(X = 6) + P(X = 7)$$

and this is true for any number of events

The Binomial Formula – Activity 2

6 **fair** six–sided dice are rolled. Can you work out the following probabilities?

NOTE: a **fair** dice is one where all numbers have EQUAL probability of being rolled

(a) Find the probability that you roll two 1s.

n =
k =
p =
1 – p =
Hence using the formula:

$P(\#ones = 2) =$

(b) Find the probability that you roll all ones

n =
k =
p =
1 – p =
Hence using the formula

$P(\#ones = 6) =$

The Binomial Formula – Activity 2

Here are some more questions if you've gotten this far.
Remember the steps we have in all the other questions.

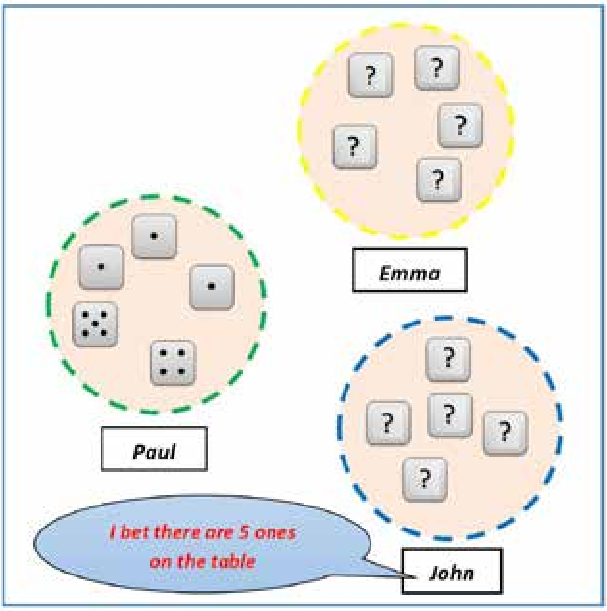
6 **fair** six–sided dice are rolled. Can you work out the following probabilities?

(c) Find the probability that you roll more than 4 ones

(d) Find the probability that you roll no ones

The Binomial Formula – Activity 3

Paul, Emma and John are playing Liar's Dice. Each has 5 dice and Paul rolls 3 ones, 1 four and 1 five. After taking turns to place bids John claims there are "5 ones on the table".



Paul then claims that John's bid is **"spot on"** (Paul thinks there are **exactly** 5 ones on the table).

Remember: Paul can only see **his** dice, he does **not** know what Emma and John have rolled!

Q: what is the probability that Paul's claim of "spot on" is correct?

Start by filling in the blanks in the following sentence

Paul has rolled _____ ones, meaning that there must be _____ ones in the remaining _____ dice on the table for there to be exactly 5 ones.

We can use the binomial formula to work out a probability for the above statement but we first need to know a few things:

VARIABLE	SYMBOL	VALUE FOR PAUL'S PROBLEM
Number of trials	n	
Number of successes wanted	k	
Probability of Success on one trial	p	
Probability of Failure on one trial	$q = 1 - p$	

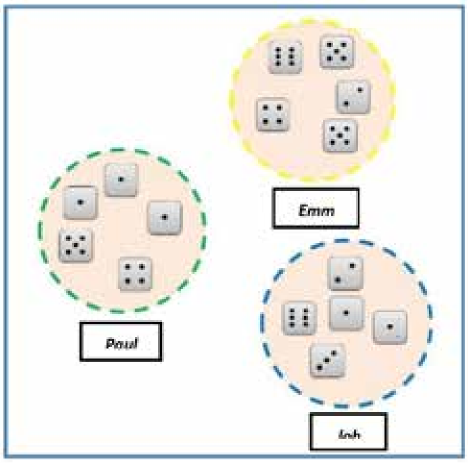
The probability that Paul's bet is exactly "spot on" is therefore:

$$P(\text{Paul is correct}) = P\,(\_\_\text{ones in }\_\_\text{dice}) =$$

Would you recommend Paul to stick with his bet? Why?

The Binomial Formula – Activity 3

Everyone reveals their dice and the results are shown in the diagram below.



Was Paul correct?

Who loses dice?

Pretend that you are Paul and that the teacher of the class comes over just before you all reveal your dice. Since the teacher cannot see any of the dice on the table, what would their answer be for the probability of your claim of "**spot on**" being correct?

Hint: Which variables in the table are different this time?

VARIABLE	SYMBOL	VALUE FOR TEACHER'S PROBLEM
Number of trials	n	
Number of successes wanted	k	
Probability of Success on one trial	p	
Probability of Failure on one trial	$q = 1 - p$	

$$P(\text{Paul is correct}) = P(\_\_\text{ones in } \_\_\text{dice}) =$$

Waves

Introduction

There are many different types of waves around us and while they may share similar properties, they do have several defining features that enable us to distinguish them from one another. For instance, mechanical waves transport their energy through a medium causing particles to oscillate. There are numerous examples of such waves including sound waves, water waves and even Mexican waves. In contrast **electromagnetic waves**, such as visible light or X–rays, do not require a medium to transport their energy and are therefore capable of travelling through a vacuum. Trigonometric functions, such as sine or cosine, can be used to model waves and are important functions in the field of mathematics. This workshop will focus on the concept of waves from both a physical and mathematical perspective.

Aim of Workshop

This workshop aims to introduce students to the various different types of waves that are found in nature, such as sound waves or ocean waves, whilst also developing a more intuitive understanding of the wave–like trigonometric functions sine and cosine. Furthermore, students will be made aware of the applications of these mathematical tools across a range of physics related disciplines.

Learning Outcomes

By the end of this workshop students will be able to

- Explain, in their own words, the concept of a wave
- Perform basic intuitive addition, subtraction and multiplication of trigonometric functions
- Discuss the widespread application of wave functions in various fields
- Understand the fundamental concepts of wave–particle duality and light polarisation.

Materials and Resources

Both optional: Slinkys, polarisers

Keywords

Wave
a disturbance or a variation which transfers energy from one point to another.

Polarisation
A phenomenon in which waves are limited to a particular direction of vibration e.g. polarised light causes vibrations in a single plane which we see using Polaroid sunglasses.

Superposition
A phenomenon which occurs when two waves meet and interact. Occasionally, such waves can add together to enhance the wave, whereas other times they simply cancel each other out.

Waves: Workshop Outline

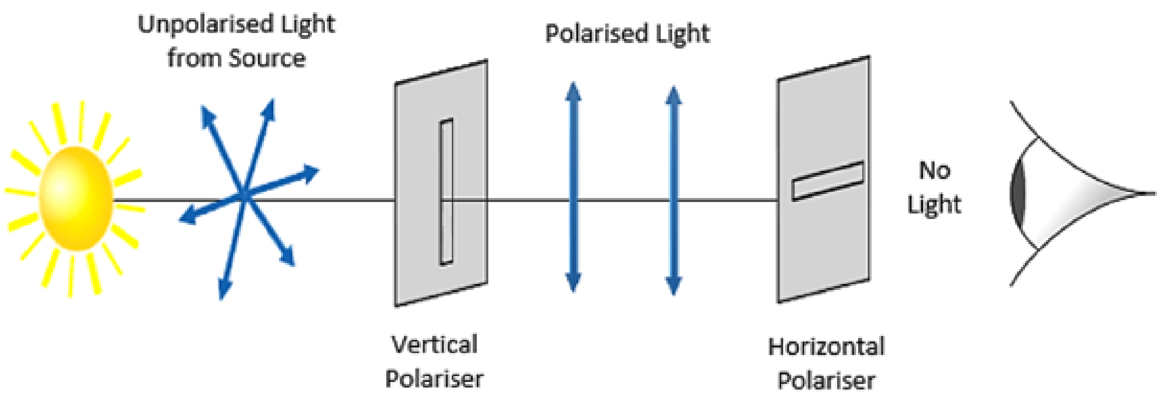
SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
10 mins (00:10)	**Introduction to the Concept of Waves**	– Whole class discussion based around the different types of waves found in nature e.g. light waves, seismic waves, ocean waves etc. – ***Note:*** *Alternatively, students can complete activity 1 whereby they are asked to list examples of waves*
5 mins (00:15)	**Mexican Wave**	– "Is it possible to make waves ourselves?" Demonstrate the Mexican wave using the students – "What is the medium?" – "What is being transferred?" – "What does a wave look like if we draw it?" (draw sine or cosine wave on the whiteboard) (Demonstrate the use of Slinkys to provide an example of a mechanical wave)
15 mins (00:30)	**Activity 2 and 3 – Mathematical Waves**	– **Activity Sheet 2:** Students attempt activity on their own before discussing the solutions as a class group – Contrast sine and cosine look like on the whiteboard – "How are they different?" – "How are they similar?" – You may like to ask the students what they think the y–axis scale is measuring. – **Activity Sheet 3:** Students attempt activity 3
15 mins (00:45)	**Superposition and Light Polarisation**	– Explain what is meant by superposition and light polarisation (see **keywords**) – Explain how a polariser works (see **Appendix– Note 1**) – **Activity Sheet 4:** Students complete activity 4 (See **Appendix– Note 2** for solution)

Waves Workshop Appendix

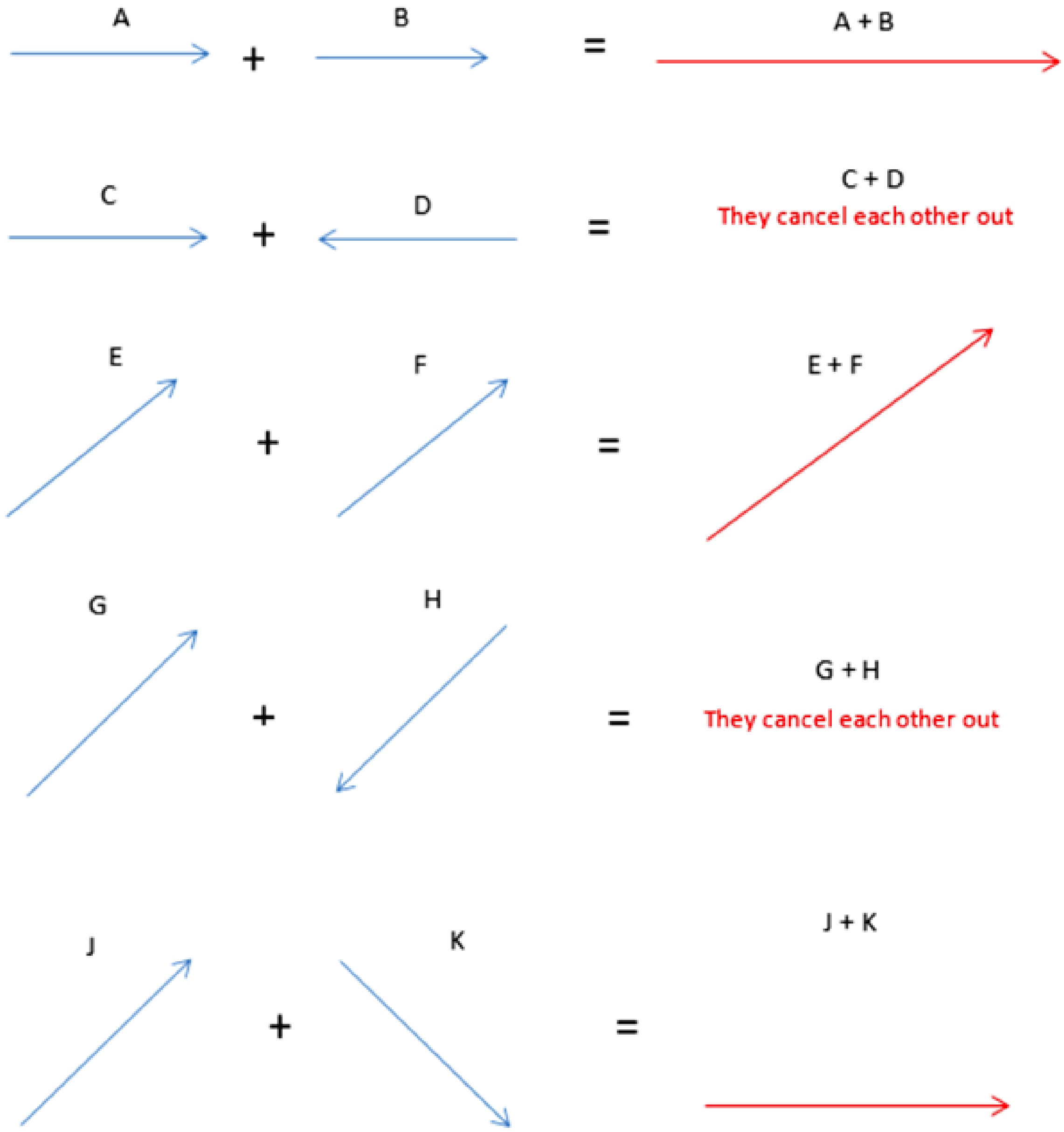
Note 1: How a polariser works

A polariser is a device that only allows light to shine in a particular direction. It can be used to convert light of undefined polarisation (i.e. travelling in lots of directions) into polarised light.

1. Take a polariser and hold it up to the light. Notice how the light only appears to be going in the one direction.
2. Take a second polariser and hold it at a 90–degree angle in front of the first polariser. Notice how no light can pass through this time. This is due to the fact that the first polariser is positioned vertically and the second one is now positioned horizontally. Thus the initial polarised light, travelling in the direction of the first polariser, cannot pass through the second.



Note 2: Superposition activity sheet solutions:



Sources and Additional Resources

http://www.physicsclassroom.com/class/waves/Lesson–1/Categories–of–Waves

https://phet.colorado.edu/

Waves: Activity Sheet 1

Physical Waves

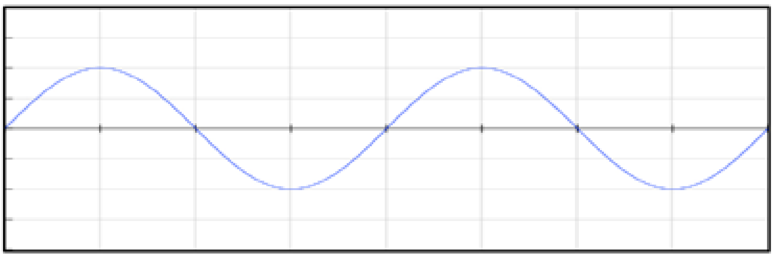
In the physical world, waves are found in many different forms. Physicists define a wave as a transfer of energy and there are many examples of waves in the world around us. See how many examples of the different types of waves you and your group can come up with...

Waves: Activity Sheet 2

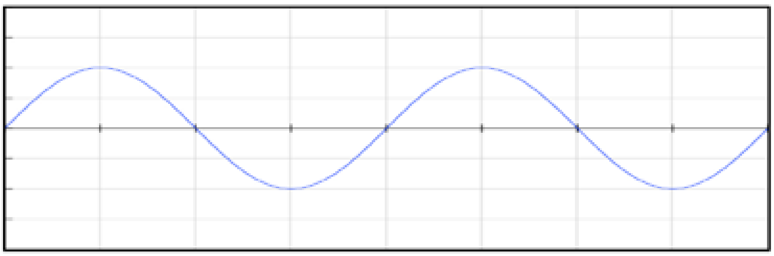
Mathematical Waves

Waves exist in the mathematical world too. In fact, waves can be manipulated using maths. Sketch, in the empty boxes, what you think a new wave would look like if you were to add the two given waves together. Use your instinct here!

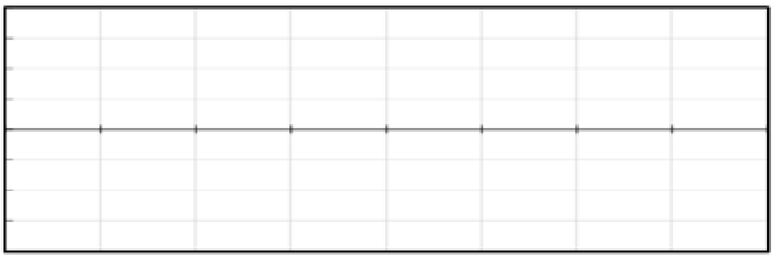
A:



B:

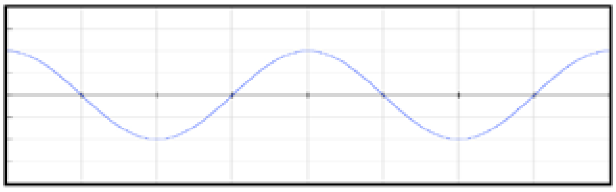


A + B:

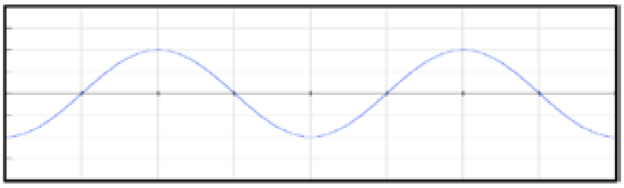


Waves: Activity Sheet 2

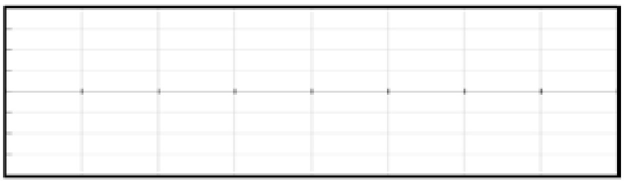
C:



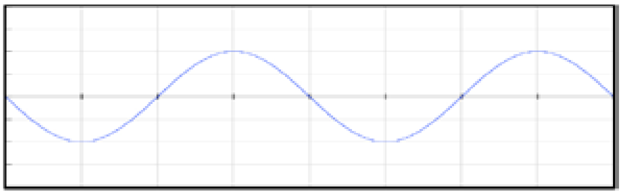
D:



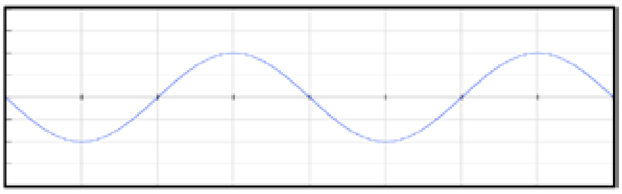
C + D:



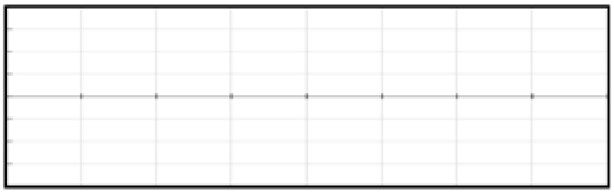
E:



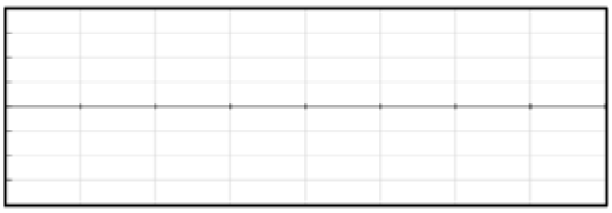
F:



E + F:



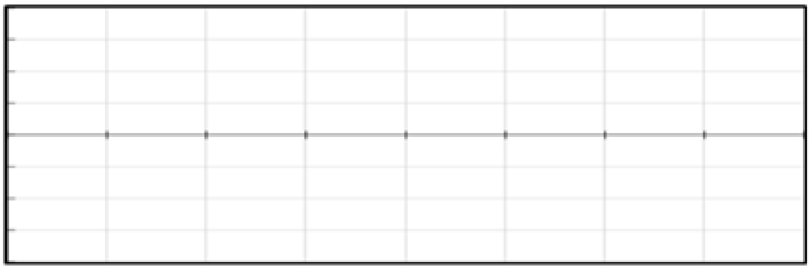
E - F:



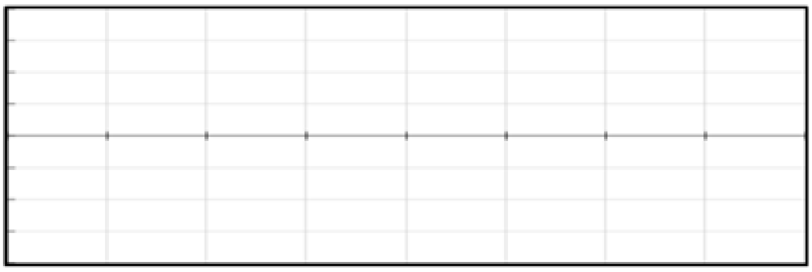
Waves: Activity Sheet 3

Now that we have seen what Sine and Cosine look like, try to figure out what it might look like if you multiply these waves by a number. If you think you don't know, guess!

$2\sin$:



$\frac{1}{2}\cos$:

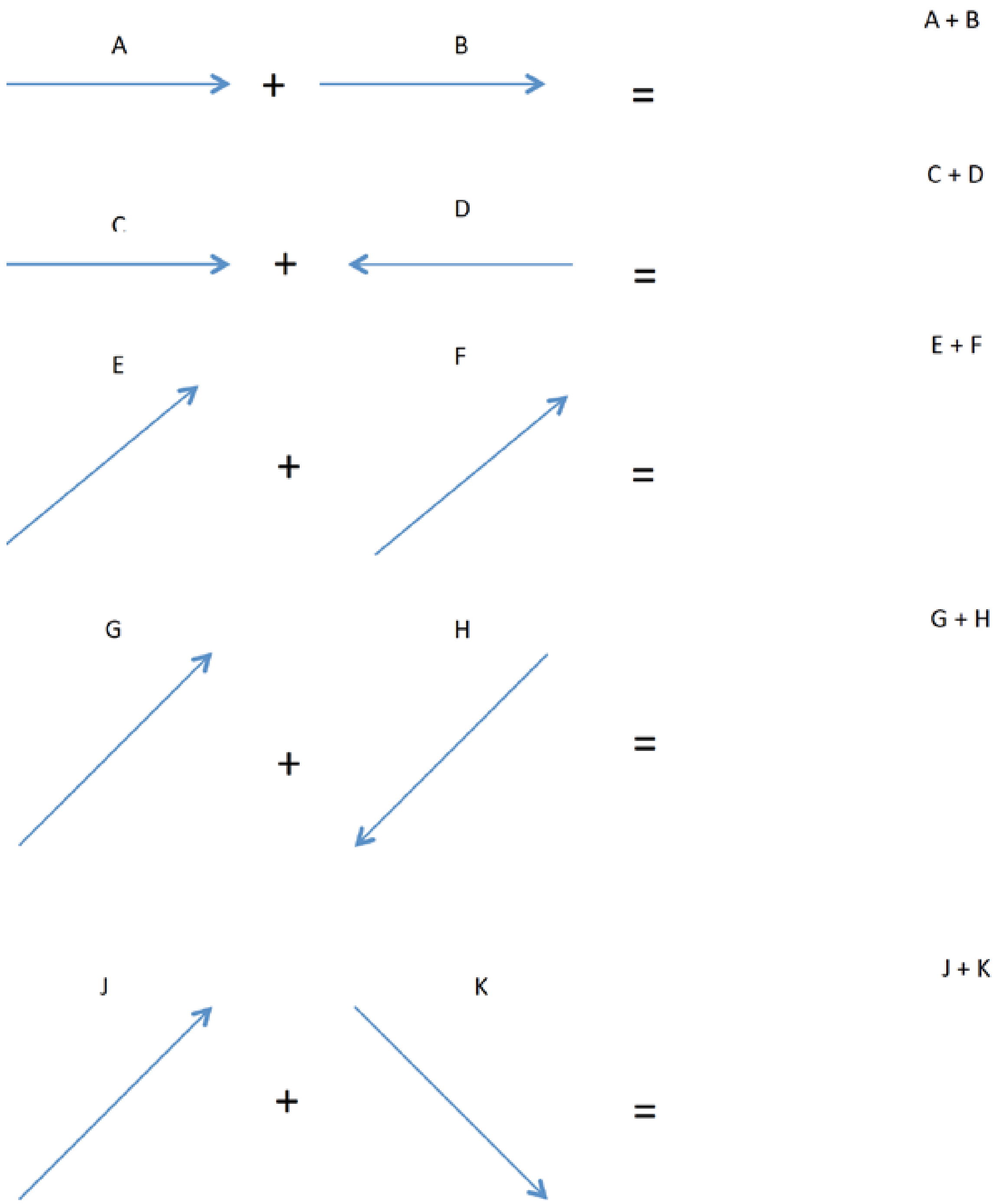


$-2\sin$:

Waves: Activity Sheet 4

Polarised Waves

In Activity 3, we learned that we can add mathematical waves together to make new waves, and that we can break bigger waves up into a collection of smaller waves. We can do this with physical waves too, for example light waves. These light waves are polarised in a certain direction. By learning how to add directions, we can understand the idea of polarisation more intuitively.



Engineering and Project Management: Developing a Wind Farm

Introduction

Mathematics is an important aspect of any construction project, whether it be a small garden shed or a large scale wind farm. Construction workers use a variety of different areas in mathematics whilst practicing their trade. For example, trigonometry is useful for calculating the pitch of a roof or the height of a windmill whilst financial maths ensures that all the project expenses are being accounted for. In fact, mathematicians and engineers regularly collaborate with one another on various construction projects and this requires a variety of skills including communication, teamwork and problem solving.

Aim of Workshop

The aim of this workshop is to engage students in the type of maths that is involved in the construction of a wind farm, whilst also developing their communication and team working skills. There are two proposed sites to host the wind farm and our engineering firm have to decide which site is most appropriate. The students will therefore be divided into two groups: Site 1 and Site 2. There will be six subgroups in each of the site groups and they will each have their own goals and tasks to work on. At the end of the workshop, the findings from each site will be accumulated and presented at "the final pitch" in a bid to win an investment.

Learning Outcomes

By the end of this workshop students will be able to:

- Recognise the importance of teamwork and communication
- Work effectively in groups on a common task
- Apply their mathematical knowledge to a real world problem.

Materials and Resources

Group 2:
Protractors, scissors, rulers, coloured paper

Group 4:
Rulers

Engineering & Project Management: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
10 mins (00:10)	**Introduction to the task and the structure of groups**	– Depending on the numbers of students, divide students into groups "Site 1" and "Site 2". (There needs to be at least 12 students in each site). – Ensure that there are 6 subgroups in each of the two site groups. – Inform students that each of the two sites must select a project manager to deliver the final pitch. This student needs to know what each subgroup is doing and be a good presenter in order to win the clients' bid. The project manager should also be a member of or join subgroup 3). – Explain to students that each subgroup must work on their own activity sheet and communicate their answers to the other specific groups (noted on their activity sheet) using a reporting sheet (included in Appendix). (It might be useful to discuss with students why such a reporting mechanism might be useful). – Additionally, each subgroup must also appoint a coordinator for their group who is responsible for delivering the report sheet to the relevant subgroup. ***Note:*** *Some of the activity sheets have different values for Site 1 and Site 2 hence they are all arranged according to the Site*
40 mins (00:50)	**Activity Sheets**	– Distribute the activity sheets so that each subgroup has a different worksheet (1 to 6). – Students work together to complete the activities on their designated worksheet. – The coordinators of the subgroups deliver the relevant figures and information to the group indicated on their activity sheet. – The project manager gathers all of the information and collaborates with his site teams on the final pitch.
10 mins (01:00)	**Pitches**	– Ask the project managers of both "Site 1" and "Site 2" to make their final pitch. – It may be useful to bring in external parties (teachers, parents, university lecturers etc.) to determine the winning site. – The group with the most ideal proposal wins the contract.

Report Sheet Site ___

To Group:

From Group:

Report:

Site 1

Engineer Group 1

Windmill Technicians

Engineers your brief is as follows: we need you to figure out the height and orientation of the windmills that are going to be in our energy farm. Some of the information you will need has already been collected by our scientists, but we need you to make sense of that data and make the best decisions you can to maximise energy output from the windmill.

You will need to work with other groups for this exercise. Assign one person to coordinate with other groups and to coordinate with the project manager.

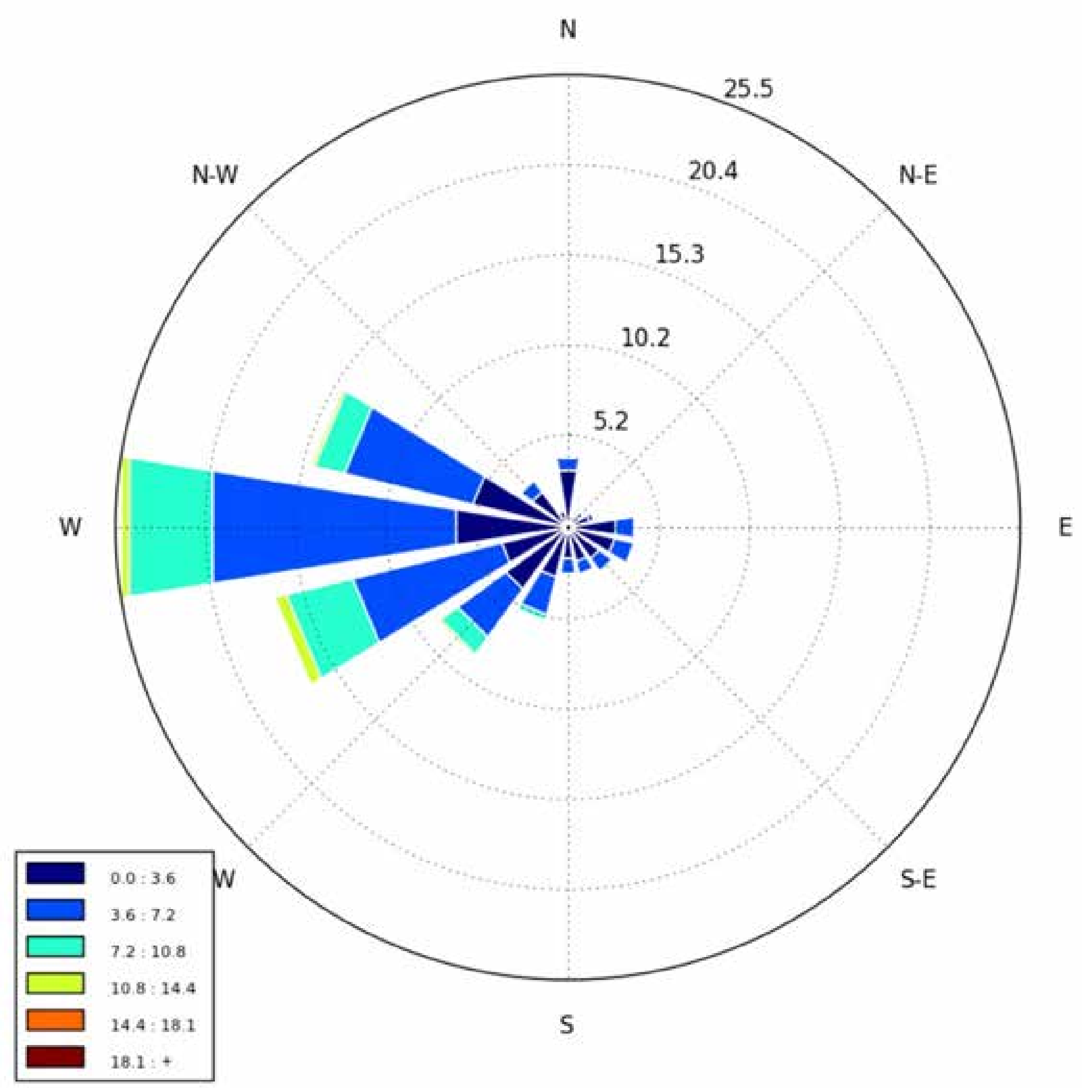
You have **three** tasks

Task 1: Your first task is to decide on the best orientation for the windmills.

To do this we will be looking at **rose plots** (graphs that measure wind speed (**in knots**) and direction). These graphs give you two different pieces of information:

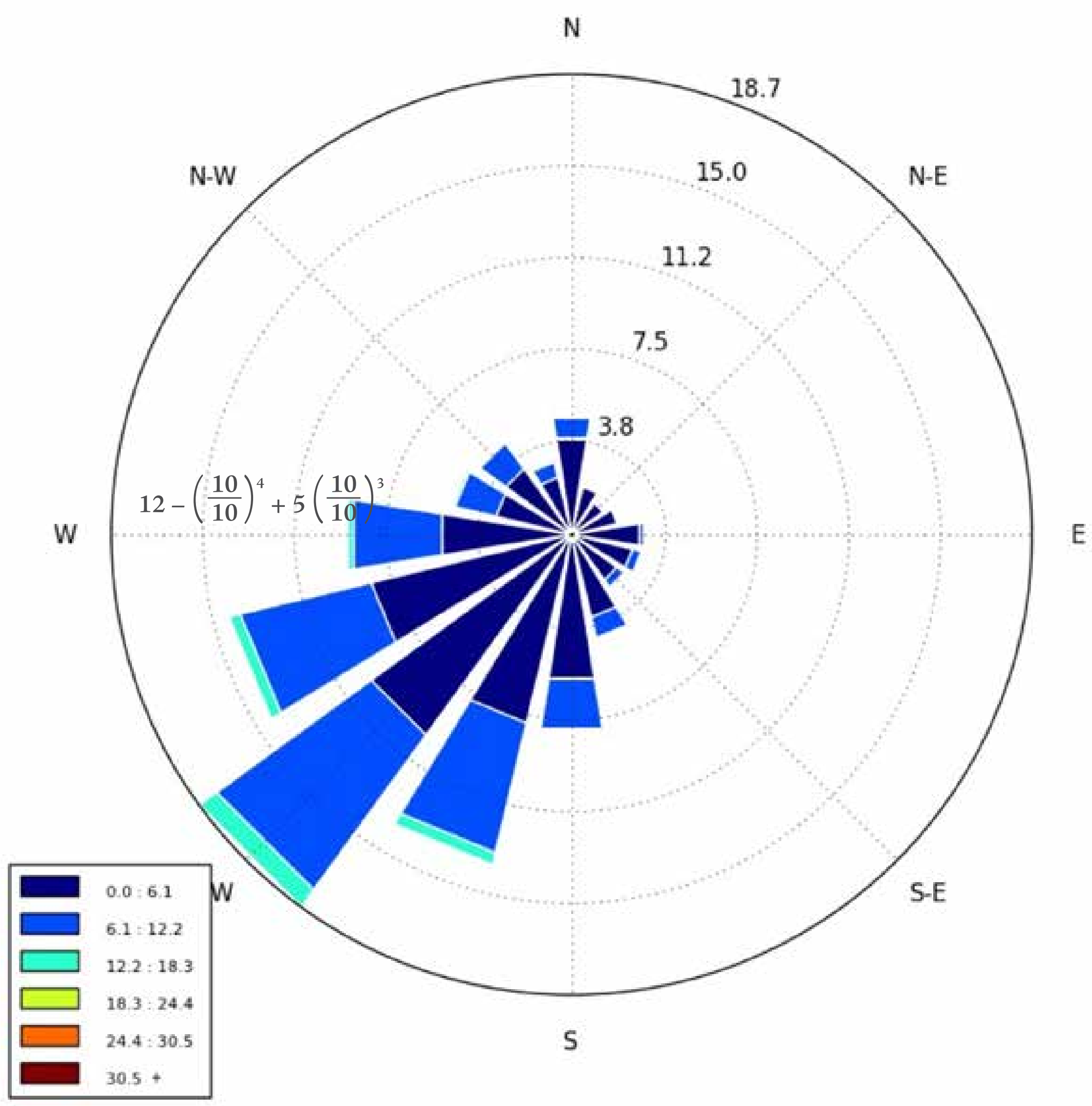
1. The **length** of the bar tells you **how much wind** came *from* that specific direction during the year. The longer the bar, the more wind came from that direction.
2. The colours in the bar tell you **how strong** the wind that came from that direction was. If a bar has lots of light blue or yellow that means the wind from that direction is very strong, if it has mostly dark blue, the wind from that direction was mostly weak (Remember, the lighter colours give the strongest wind!)

Here are two rose plots for the site, collected by Met Éireann, the first is average for the summer months, and the second is average for the winter months.



Credit http://www.niallmcmahon.com/swc_2015_notes.html

Summer Wind Data (in Knots)



Credit http://www.niallmcmahon.com/swc_2015_notes.html

Winter Wind Data (in Knots)

Given these graphs, decide what direction the windmills should face in order to maximise the amount of wind they will receive.

When you have this done please report, using the report card, to **Group 2** with your findings before moving on to the next activity.

Task 2: Find the most suitable height for the windmill

Our scientists have found that the relationship between wind speed and height from the ground is

$$Wind\ Speed = 12 - \left(\frac{Height}{10}\right)^4 + 5\left(\frac{Height}{10}\right)^3$$

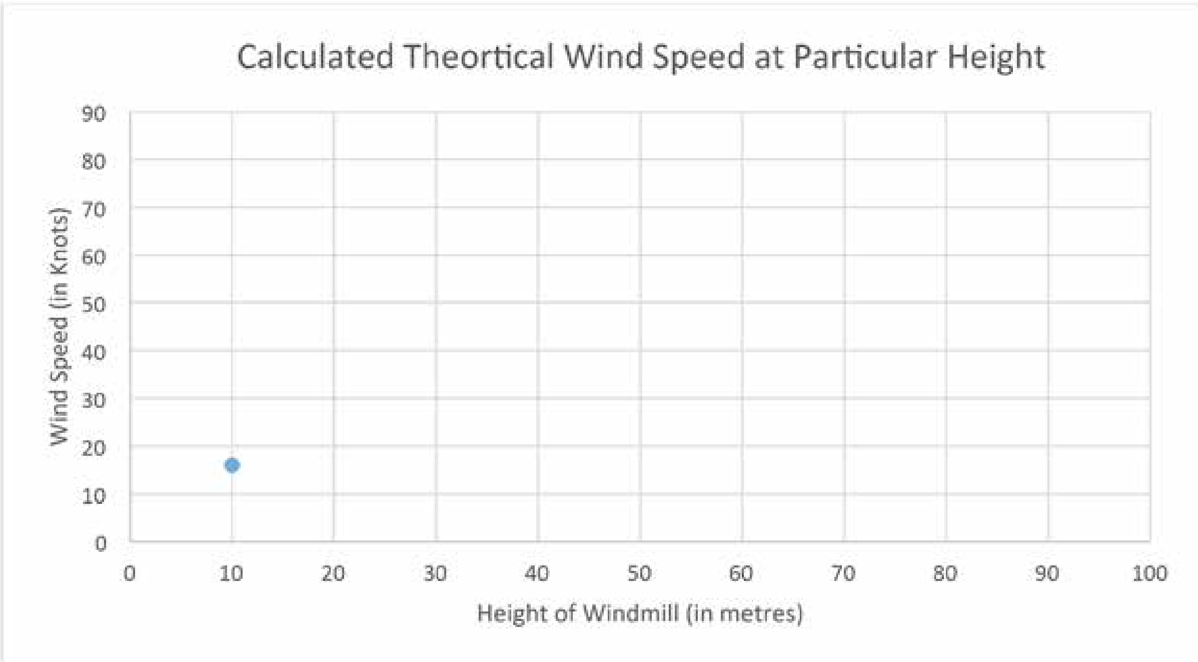
To get a sense of what windmill height will capture the best wind speed we need you to check different values for height and find the wind speed at that particular height. The first is done already:

HEIGHT (M)	WIND SPEED EQUATION	(WORK OUT FRACTIONS)	(WORK OUT POWERS)	(SIMPLIFY)	VALUES (HEIGHT, WIND SPEED)
10		$12 - 1^4 + 5(1^3)$	$12 - 1 + 5(1)$	16	(10,16)
20					
30					
40					
50					

What's our best estimate for the height of the windmill?

Plot your findings to try and find a better answer.

For each of the values we found, record a point on the graph that corresponds with the x and y axis. The first one is done as an example:



What's our best choice for the height of the windmill now?

(Is this different to our previous estimate?)

Task 3. Find out the maximum height that we can safely make the windmills.

For this you need to go to the Safety Engineers (**Group 6**) and ask for their findings.

Given all of the information you have gathered please choose at what height and orientation you have decided the windmills should be. Record your findings on a separate sheet and give to your project manager.

Congratulations, engineers! You have successfully completed this exercise. If you're done, see if you can assist other teams on your site

Site 1

Engineer Group 2

Spatial Engineers

Engineers, your goal is to find the maximum number of windmills that can be safely installed into the site.

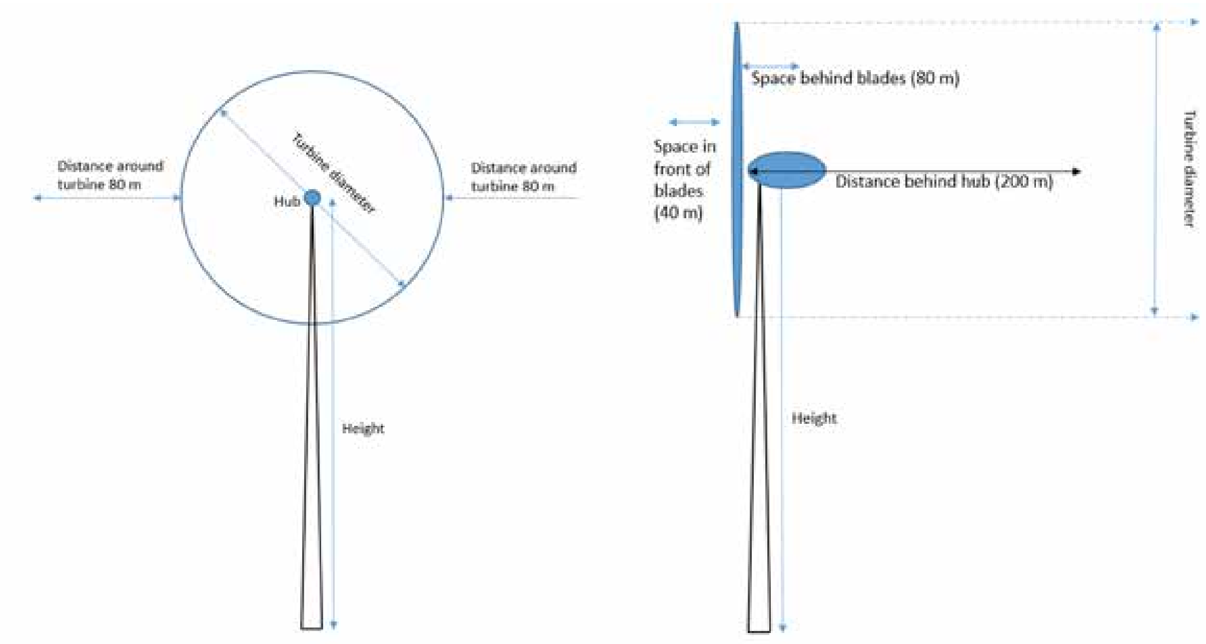
You will need to work with other groups for this exercise. Assign one person to coordinate with other groups and to coordinate with the project manager.

This exercise is split in two parts. In the first part you will calculate the **space** that is required to be left clear around each turbine. Then, using information provided by **Group 1**, you will decide the maximum number of turbines that we can safely have on the wind farm.

Task 1: Assess the space needed for each turbine.

For safety reasons, there are regulations about how close each turbine can be to another. However, the space that is needed around a turbine is determined by its size. The size of the turbine is its turbine diameter. Windmills with large turbine diameter are built to be taller. There are two graphs below:

- **Graph 1 shows:**
  - The typical height of a windmill given the turbine diameter.
- **Graph 2 shows:**
  - The amount of space that should be left clear in front of turbine's blades,
  - The space that should be left clear behind the turbine's blades.
  - The space that should be left clear directly behind the hub centre. The hub centre is at the centre of where the turbine blades meet



To calculate the clear area needed around each turbine:

1. Consider the two schematics shown above. It shows that:
   a. 80 m must be left clear at every point behind the blades
   b. 40 m must be left clear at every point in front of the blades.
   c. 200 m must be left clear behind the centre of the hub.
   d. 80 m must be left clear around the turbine blades.

2. Consult with each other to design a shape that represents the area required around each turbine. You are welcome to come up with more than one design if you like as long as they satisfy the safety regulations above. There is no one right answer. Be creative!

Task 2: Find the maximum number of turbines possible to place onto the site

Now that you know the dimensions of the shapes, you'll need to figure out how many of each shape you can fit onto the site.

Here's how to proceed. For each shape you have designed:

1. Make lots of cut–outs of that shape. Make sure that they are the dimensions that you decided in the last part.
2. The scaling is 1 cm to 100 m.
3. Consult with the engineers in **Group 1** to find out the direction in which the turbine should be pointing.
4. Use this direction to orient the shape on the site (picture given below).
5. Fit as many of the shapes as possible. (Be careful though to keep the wind mill pointed in the correct way).
6. Report your final answer to the engineers in **Group 5** so they can complete their exercise.

Fill out one project card and hand it in to your project manager.

Congratulations, engineers! You have successfully completed this exercise. If you're done, see if you can assist other teams on your site.

Site 1

Engineer Group 3

Financial Analysts

Engineers, your goal is to analyse the cost and profit from the site. You will need to work with other groups so appoint somebody from your group to be the coordinator.

The project manager, who is also a member of your group, will have this task of representing the whole site. Ensure that your project manager is involved and updated at every stage so that they can make the best case for this windfarm.

Task 1: Find the costs of building a wind farm

There are a lot of costs when it comes to building a wind farm. In fact, we will not be able to consider all of them but we will consider the most important or obvious ones.
We can think of the costs under the following headings:

1. The Site
2. The Components
3. Maintenance
4. The Engineer

1. The Site:

It costs money to purchase the site. Consult with the Land Surveyors from Group 2 to get the cost of purchasing the land:

It also costs money to prepare the land. This means that you will have to make clear large rocks, big bushes and shrubs and level the land. This will cost a flat rate of €5,000.

What is the total cost of the site including site preparation?

2. The Components:

It will cost to build the windmills for the site. Each windmill has

a. 1 tower

b. 1 hub

c. 3 blades

The following quotes for the windmill components have been given from 3 different companies Irish Wind, Storm Chasers and Propellers. We are new customers but would hope to become regular customers.

Irish Wind is located in Dublin, roughly 120 km from the wind farm. It has been established and trading in energy systems since 1980.

ITEM	QUOTE FOR NEW CUSTOMERS (€)	QUOTE FOR REGULAR CUSTOMER (€)	EXPECTED LIFETIME (YEARS)
Turbine blade	1,400	1,300	3
Hub	15,000	15,000	6
Tower	20,000	18,000	12

Storm Chasers is located in London, roughly 250 km from the site. It was established in 1995. It is reputed to have supplied many units to several American and British wind energy projects.

ITEM	QUOTE FOR NEW CUSTOMERS (€)	QUOTE FOR REGULAR CUSTOMER (€)	EXPECTED LIFETIME (YEARS)
Turbine blade	1,200	1,000	3
Hub	20,000	18,000	8
Tower	18,000	15,000	12

Propellers Inc. is located in Galway, roughly 50 km from the site. It was established in 2002 and is completely Irish owned.

ITEM	QUOTE FOR NEW CUSTOMERS (€)	QUOTE FOR REGULAR CUSTOMER (€)	EXPECTED LIFETIME (YEARS)
Turbine blade	2,000	1,500	5
Hub	22,000	18,000	10
Tower	21,000	17,000	15

Consider the quotes you have been given for the components. Which company would be most beneficial (either in the short term or in the long term or both) and would maximise the profits? Be prepared to explain your reasoning and communicate your findings with the Project Manager. There are no wrong answers, but there are optimum situations.

How much will the components cost in total (from your chosen supplier)?

3. Maintenance:

There are many things that need to be maintained on a wind farm.

- The windmill itself needs to be cleaned regularly to ensure that it is working efficiently.
- If any of the parts fails, it will need to be replaced.
- Even if it does not fail, there will be a recommended lifetime for each component of the windmill. When it has been operating for as long as its recommended lifetime, then it will need to be replaced even if it is still functional.

To calculate the replacement of parts. Consider the lifetime of the components that you decided on and split the cost evenly over the years. For example, if the lifetime of a component is 5 years and costs €5,000 you will assign €1,000 for it a year.

Can you calculate the replacement costs for each of the components from your chosen supplier?

3 Blades:

1 Tower:

1 Hub:

4. The Engineer:

An engineer will be contracted to make regular visits to the wind farm to ensure that all windmills are running as expected. The income they generate from this will be considered their regular salary.

Below there are three candidates. You are provided with some information about their qualifications, their work experience (if they have any) and the salary the candidate would get if they are hired. Discuss among each other and decide who you would hire.

Michael O'Brien

1. Finished his PhD in UCD last year in Wind Energy Systems
2. 5 years working experience in off–shore wind off the coast of Denmark
3. Expected salary: €50,000 per year

Linda McNally

1. Degree in Electrical Engineering with Management
2. Worked as project manager with the Sustainable Energy Authority Ireland on several wind energy projects and has experience in research over the last 10 years
3. Expected salary: €60,000 per year

Jordan Hoffmann

1. Graduated with a Master's degree in alternative energy systems this year
2. No work experience
3. Expected salary: €34,000 per year

Based on the above information, which candidate would you hire and why?

Task 2: Calculate the net profit and prepare your presentation

Consult with the engineers from Group 5 to get the income generated from the wind farm.

Table 1: This table will help you put together all the costs (for one year) that you found from your first task.

ITEM / SERVICE	COSTS
Purchase of site	
Preparation of site	
Cost of components from chosen supplier	
Replacement of parts	
Engineer's salary	
Total costs:	

Comparing the total income generated from the wind farm with the total costs to build the wind farm, can we operate effectively as a business? Explain your reasoning

Congratulations, engineers! You have successfully completed this exercise. If you're done, see if you can assist other teams on your site.

Site 1

Engineer Group 4

Land Surveyors

Surveyors, your task is to calculate the cost of purchasing the site for the wind farm.
Choose one person who will coordinate with other groups and the project manager. You will need to calculate how big the site is. Then using this information and with the help of the other engineers, you will be able to calculate the total cost of the land. Let's get started!

Task 1: Calculate the surface area of the site.

The shape of the site is given below. The scaling is 1cm = 100 m. This means that every centimetre that your ruler measures, is the same as 0.1 km in the real world.

1. Using a straight edge or ruler, split the above shape into rectangles and triangles.
2. Use Tables 1 and 2 on the next page to calculate the area of each of the shapes, remembering to convert each of your measurements to kilometres.

Table 1: This table will help you to calculate the area of all the triangles.

TRIANGLE	MEASURE THE LENGTH OF THE BASE, B (in km)	MEASURE PERPENDICULAR WIDTH, W (in km)	AREA: CALCULATE 0.5 x B x w (in km²)
1.			
2.			
3.			
4.			
5.			
6.			
7			
8.			
9.			
10.			
11.			
12.			
CALCULATE THE TOTAL AREA OF THE TRIANGLES:			

Table 2: This table will help to calculate the area of the all the rectangles.

RECTANGLE	MEASURE LENGTH, L (in km)	MEASURE WIDTH, W (in km)	AREA: CALCULATE l x W (in km2)
1.			
2.			
3.			
4.			
5.			
6.			
7			
8.			
9.			
10.			
11.			
12.			
13.			
14.			
15.			
16.			
17.			
18.			
19.			
20.			
21.			
22.			
23.			
24.			
25.			
26.			
27.			
28.			
29.			
30.			
31.			
32.			
33.			
34.			
CALCULATE THE TOTAL AREA OF THE RECTANGLES:			

Task 2: Calculate the price of the land:

Now that you know how large the area of the land is, you can proceed to calculate the price of the land:

1. The unit price is €27,000 per km2.
2. Fill in Table 4 to find your final answer.
3. Give your final answers to the engineers in Group 3 so they can complete their challenge.

Table 4: This table will help you to find the total cost of purchasing this piece of land.

SHAPE	PUT IN TOTAL AREAS FROM TABLES 1 AND 2	MULTIPLY BY THE UNIT PRICE
Triangle		
Rectangle		
Totals	Total area=	Total price=

Prepare to report to your Project Manager using the report card.

Congratulations, engineers! You have successfully completed this exercise. If you're done, see if you can assist other teams on your site.

Site 1

Engineer Group 5

Electrical Engineers

Engineers, your brief is as follows, we need you to work out the maximum wattage the windmills can output and then work with the Financial Analysts (group 3) to calculate our possible profits from the windfarm's energy output.

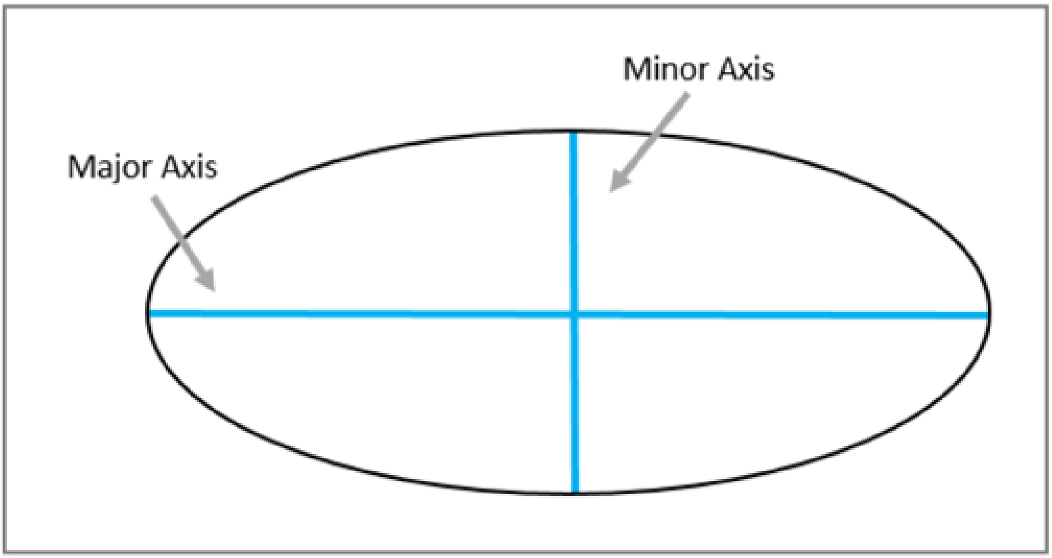
Your first task is to figure out the maximum wattage the windmills can create in order to tell the Safety Engineers (**Group 6**).

The way we'll do this is by finding out first how fast the blades will spin when the wind is at its highest. Then find out how many watts this speed will give us.

Task 1: Find the speed of the blades at maximum wind speed

To find the speed of the blades we first need to find the area of the blades and use what we know about the relationship between the area and the speed of the blades to find their maximum speed.

Our first step is to find the area of the blades. The blades of a windmill are essentially shaped like an ellipse (which is the term for a squashed circle or an oval). The information we need to know about an ellipse are the lengths of its minor axes and major axes (which you can see in the diagram below).



The formula for the area of an ellipse is:

$$A = \pi ab$$

Where

A	AREA OF ELLIPSE
a	**The length of the semi–minor axis (half the length of the minor axis)**
b	**The length of the semi–major axis (half the length of the major axis)**

Our windmill blades are **4.2m** wide at their widest part and they are **18.8m** long.

Using this can you find out how much area one blade has? How about all three blades?

Now we need to find out how fast the blades go at the maximum wind speed. Because the blades aren't moving in a straight line (they are rotating around the hub), this speed is given in revolutions per hour.

Our mathematicians have found that relationship of:

$$\omega = \frac{A * v * \sin(\theta)}{8.0498}$$

Where

ω	**The angular velocity (speed the blade is turning) of the blades (in revolutions per hour)**
A	**The area of all three blades (in metres squared)**
v	**The speed of the wind (in knots)**
θ	**The angle of the blades (in degrees)**

We've already found A and we know that the blades will be at an angle of 15 degrees, but we still need to know the max speed of the wind which you can get from **Group 1**, the Windmill Technicians.

Calculate the angular velocity of the blades:

Once you have the maximum angular velocity, all we need is to convert this to a maximum wattage. We know that one revolution per hour will create 0.24 MW (megawatts) of power with these turbines.

Use the figure you found above for angular velocity (ω) to work out the maximum wattage output.

Task 2: Calculating the income generated by the wind farm

For this task, you will be working with the Financial Analysts, **Group 3**, to work out how much money the windfarm will make on average over the course of the year.

Firstly, you will need to figure out how much money one windmill makes per year. Then ask **Group 2** how many windmills we will have in order to calculate the windfarm's income per year.
In order to find the amount of money the windmill makes, we need to find out how much energy it will produce.

We know how much power the windmills produce in megawatts from Task 1. We need to convert this to megawatt hours (a unit of energy) by multiplying the power by the amount of time the windmill is producing energy each year.

The windmills will be turned on 12 hours per day, every day of the year, except for Christmas Day, and for one day every 13 weeks for maintenance work.

This is enough information to calculate how many hours the windmill will be on for in a year.

Hours:

Now we just need to find the value, in euro, of that energy. We know that the suppliers of electricity in Ireland buy energy as described in **table 1.**

We will sell energy to each supplier equally (i.e. **25%** of the energy we produce will go to each supplier)

This is enough information to find our annual income, please report to **Group 3** with your findings once you have finished. **Note:** 1 megawatt hour = 1000 kilowatt hour

Table 1: Showing the rate of energy purchased by various suppliers

COMPANY	RATE PURCHASED	PROJECTED AMOUNT FROM EACH COMPANY
Electric Ireland	**0.032c per kilowatt hour**	
Pinergy	**33c per megawatt hour**	
Airtricity	**0.034c per kilowatt hour**	
Energia	**€1.5 per 5 megawatt hours**	
Annual income total:		

Site 1

Engineer Group 6

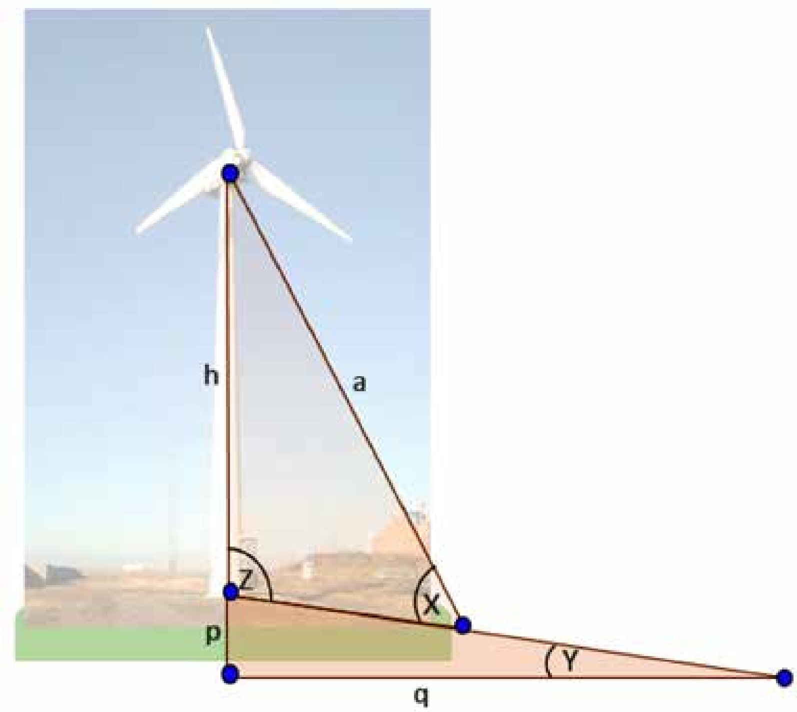
Safety Technicians

Engineers, your brief is as follows: we need you to figure out the maximum safe height of the windmills that are going to be in our energy farm. Given some guidelines about how the windmill has to be anchored to the ground, you will then need to design a safe electric grid to provide energy to the local community.

Your first job is to find the maximum allowable height for the windmills. Under E.U. regulations, the maximum angle **relative to the ground** our anchors are allowed be placed is **65°**. Also, the anchors must be attached to the top of the windmill. Lastly, the anchors are limited to being **55m** long.

This is enough information to find the maximum height we can build the windmills on flat ground. However, the windmills will all be placed on the top of artificial hills, where the height drops by **1m** for every **10m** you go out.

What you'll have to do is it put all of this information onto the diagram below:



Where,

h	**Height of windmill**
a	**Length of anchor**
p	**Height of hill**
q	**Width of hill**
x	**Angle of anchor relative to the ground**
y	**Angle of elevation of hill**
z	**Angle between windmill and ground**

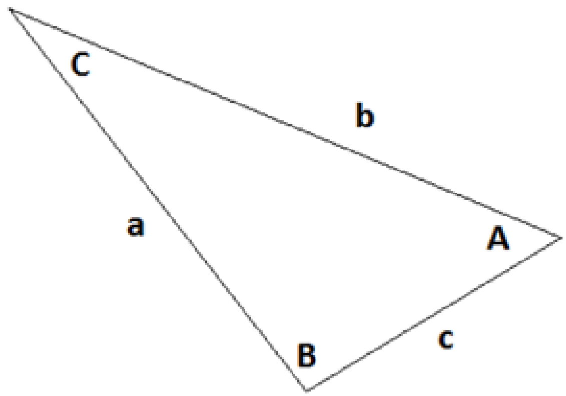
We're looking to find h, but we do this in a number of steps:

1. **Find the slope of the base triangle by finding the rise over the run,** $\frac{p}{q}$.

2. **The slope (m) = tan(Y). Find Y.**

3. **Find the size of the angle z.**

Next we're going to need to use the sine rule, which states that if you have a triangle like the one below:



Then:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

That means that if you knew, say, what a, b and $\sin(B)$ were, you could find $\sin(A)$ using algebra, i.e.

$$\sin(A) = \frac{a * \sin(B)}{\mathrm{b}}$$

So we can now go on to our last steps:

4. Use the sine rule to find the height of the windmill, h.

Once you have found this, report to engineer **Group 1** with your findings.

Task 2: Design the local electric grid

In any electrical system, engineers must be careful of overloads (where an excessive amount of current in the wire causes heat and possible damage to equipment).

For your next task, you are going to have to design the local electric grid to ensure that the likelihood of overloads in the grid is acceptably low, to do this you will first have to find out how many houses you can connect successively for different wattages.

The local council have decided that the probability of an overload during the course of a connections lifetime must be below 0.05 for it to be safe. Our scientists have found that the probability of an overload for n connected houses, with appropriate wattage w is:

$$P\ (overload) = \frac{n}{n + 100 - w}$$

Where,

N	**Number of houses connected**
W	**Wattage (in MW or megawatts)**
p	**Probability of an overload in the connection's lifetime**

Given this, how many houses could we safely connect for these wattages? We've done the first one to get you started.

The first calculation for n = 1 is:

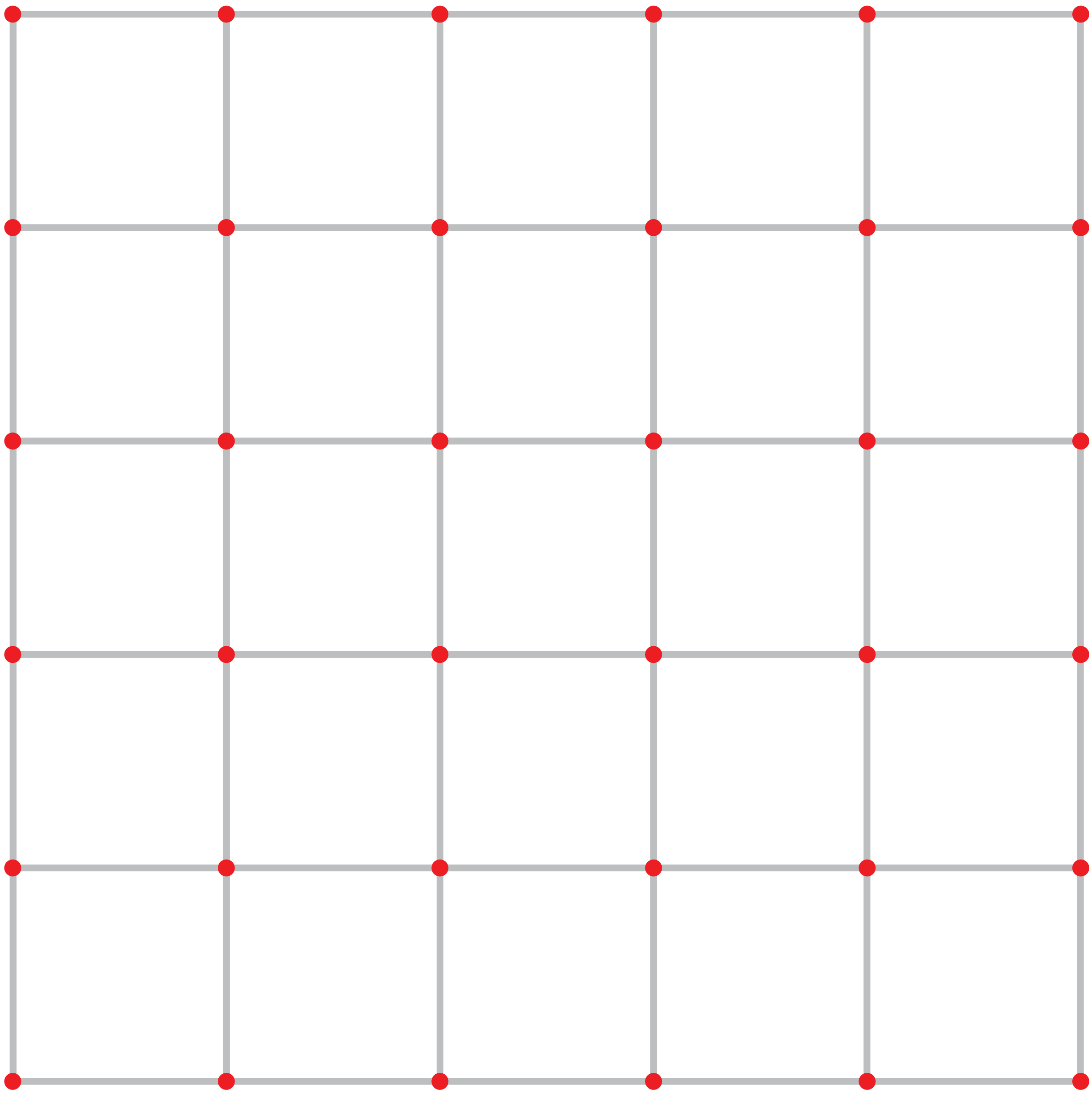
$$\frac{1}{1 + 100 - 10} = 0.011$$

Wattage	Prob. For N=1	Prob. For N=2	Prob. For N=3	Prob. For N=4	Prob. For N=5
10MW	**0.011**	**0.022**	**0.032**	**0.043**	**0.053 (too big)**
20MW					
30MW					
40MW					
50MW					

Now ask the Electrical Engineers **(Group 5)** what the maximum wattage the windmills can output and find the corresponding maximum number of houses we can connect.

Then you need to place the smallest number of transformers on this grid so that you can provide energy to all the houses.

Each red dot on the grid represents a house and you can place a transformer on any of the red dots, this provides energy to that house but doesn't use up one of the connections.



Congratulations, engineers! You have successfully completed this exercise. If you're done, see if you can assist other teams on your site.

Site 2

Engineer Group 1

Windmill Technicians

Engineers, your brief is as follows: we need you to decide on the height and orientation of the windmills that are going to be in our energy farm, some of the information you will need has already been collected by our scientists but we need you to make sense of that data, and make the best decisions you can to maximise energy output from the windmill.

You will need to work with other groups for this exercise. Assign one person to coordinate with other groups and the project manager.

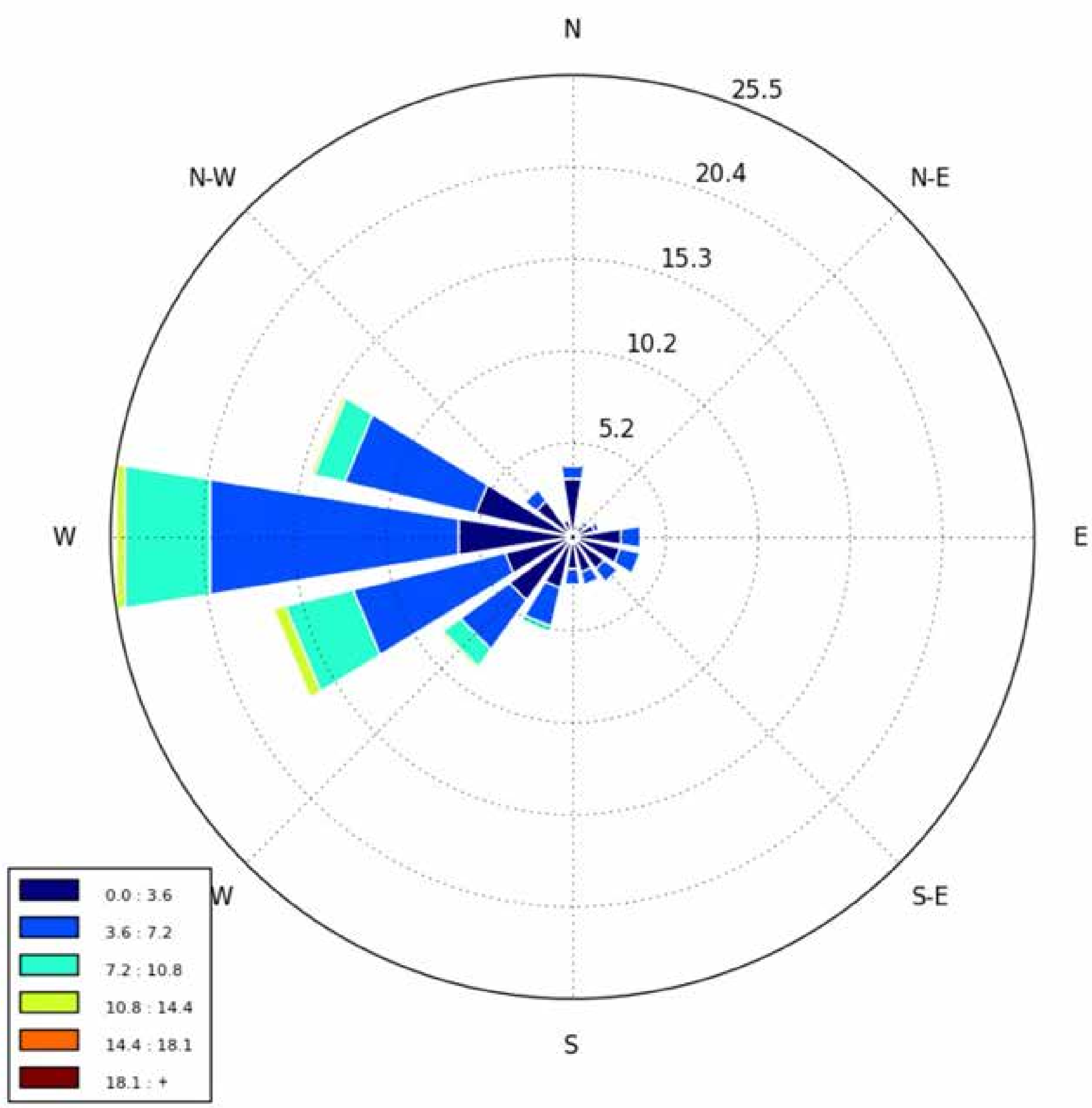
You have **three** tasks.

Task 1: Your first task is to figure out which orientation is best for the windmills.

To do this we will be looking at rose plots (graphs that measure wind speed (in knots) and direction). These graphs give you two different pieces of information:

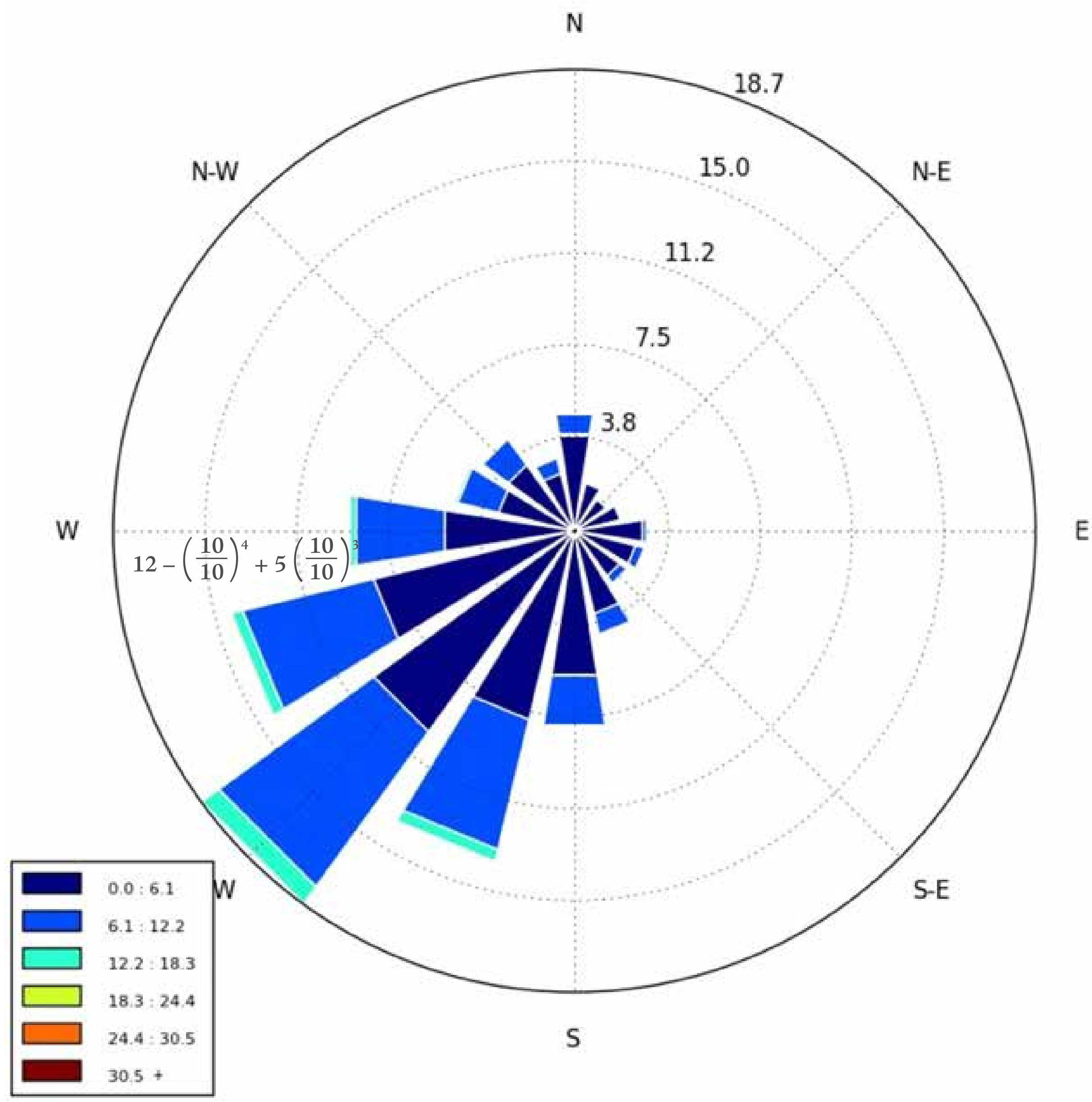
1. The **length** of the bar tells you **how much wind** came *from* that specific direction during the year. The longer the bar, the more wind came from that direction.
2. The **colours** in the bar tell you **how strong** the wind that came from that direction was. If a bar has lots of light blue or yellow, that means the wind from that direction is very strong. If it has mostly dark blue, the wind from that direction was mostly weak (Remember, the lighter colours give the strongest wind!)

Here are two rose plots for the site, collected by Met Éireann, the first is average for the summer months, and the second is average for the winter months.



Credit http://www.niallmcmahon.com/swc_2015_notes.html

Summer Wind Data (in Knots)



Credit http://www.niallmcmahon.com/swc_2015_notes.html

Summer Wind Data (in Knots)

Given these graphs, we need you to decide what direction the windmills should face in order to maximise the amount of wind they will receive.

When you have this done, please report, using the report card, to Group 2 with your findings before moving on to the next activity.

Task 2: Find the most suitable height for the windmill

Our scientists have found that the relationship between wind speed and height from the ground is

$$Wind\ Speed = 12 - \left(\frac{Height}{10}\right)^4 + 5\left(\frac{Height}{10}\right)^3$$

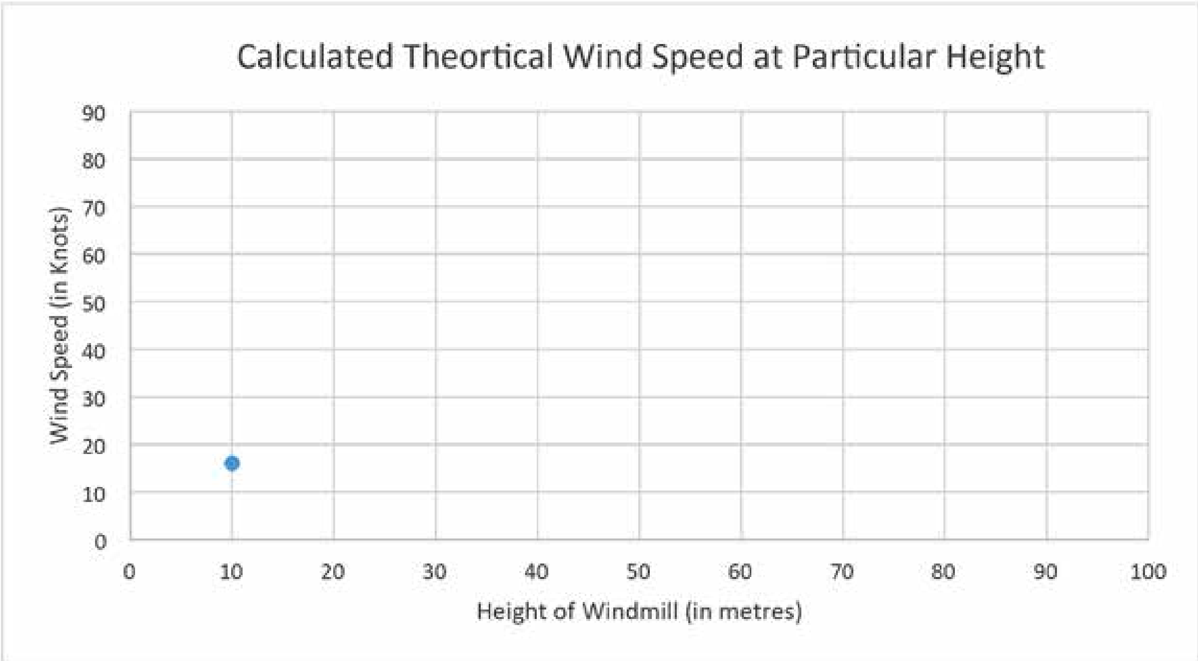
To get a sense of what height will produce the best wind speed we need you to check different values for height and to find the wind speed at that particular height. The first is done already: we've done one of them for you already:

HEIGHT (M)	WIND SPEED EQUATION	(WORK OUT FRACTIONS)	(WORK OUT POWERS)	(SIMPLIFY)	VALUES (HEIGHT, WIND SPEED)
10		$12-1^4+5(1^3)$	$12-1+5(1)$	16	(10,16)
20					
30					
40					
50					

What's our best estimate for the height of the windmill?

Let's plot our findings to try and find a better answer.

For each of the values we found, we will plot a point on the graph that corresponds with the x and y axis. The first one is done as an example:



What's our best choice for the height of the windmill now?

(Is this different to our previous estimate?)

Task 3. Find out the maximum height that we can safely make the windmills.

For this you need to go to the Safety Engineers **(Group 6)** and ask for their findings.
Given all of the information you have gathered please choose at what height and orientation you have decided the windmills should be.

Record your findings on a separate sheet and give to your project manager.

Congratulations, engineers! You have successfully completed this exercise. If you're done, see if you can assist other teams on your site.

Site 2

Engineer Group 2

Spatial Engineers

Engineers, your goal is to find the maximum number of windmills that can be safely installed into the site.

You will need to work with other groups for this exercise. Assign one person to coordinate with other groups and the project manager.

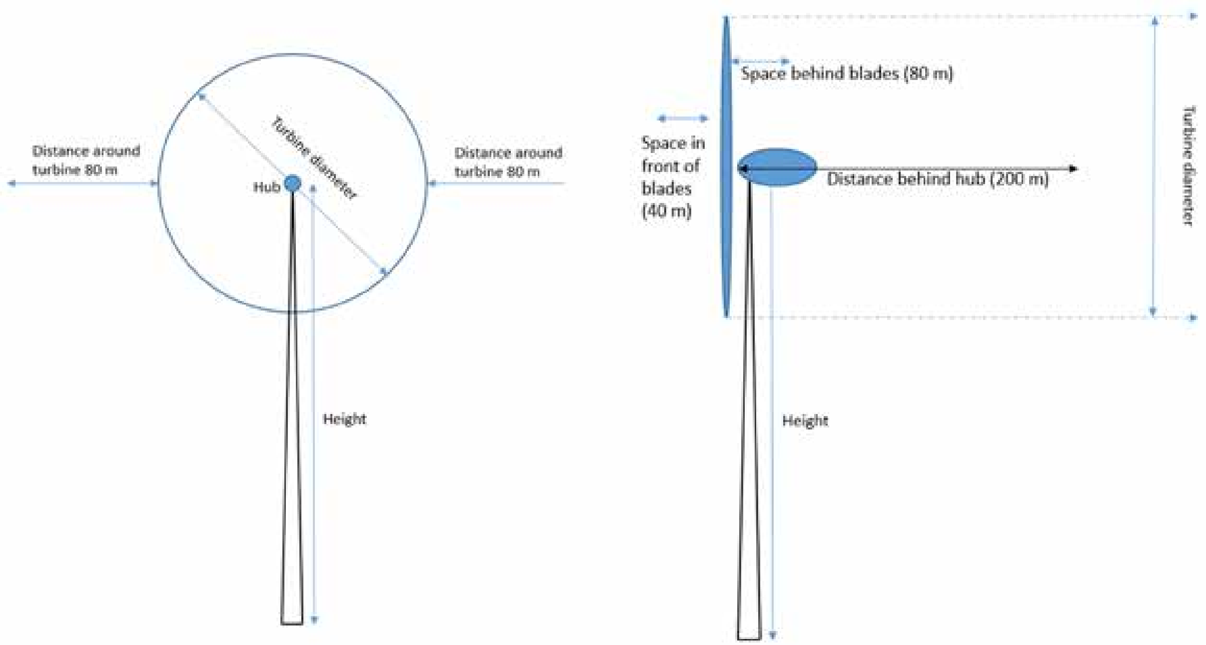
This exercise is split in two parts. In the first part you will calculate the **space** that is required to be left clear around each turbine. Then, using information provided by **Group 1**, you will decide the maximum number of turbines that we can safely have on the wind farm.

Task 1: Assess the space needed for each turbine.

For safety reasons, there are regulations about how close each turbine can be to another. However, the space that is needed around a turbine is determined by its size. The size of the turbine is its turbine diameter. Windmills with large turbine diameter are built to be taller. There are two graphs below:

- **Graph 1 shows:**
  - The typical height of a windmill given the turbine diameter.
- **Graph 2 shows:**
  - The amount of space that should be left clear in front of turbine's blades.
  - The space that should be left clear behind the turbine's blades.

The space that should be left clear directly behind the hub centre. The hub centre is at the centre of where the turbine blades meet.



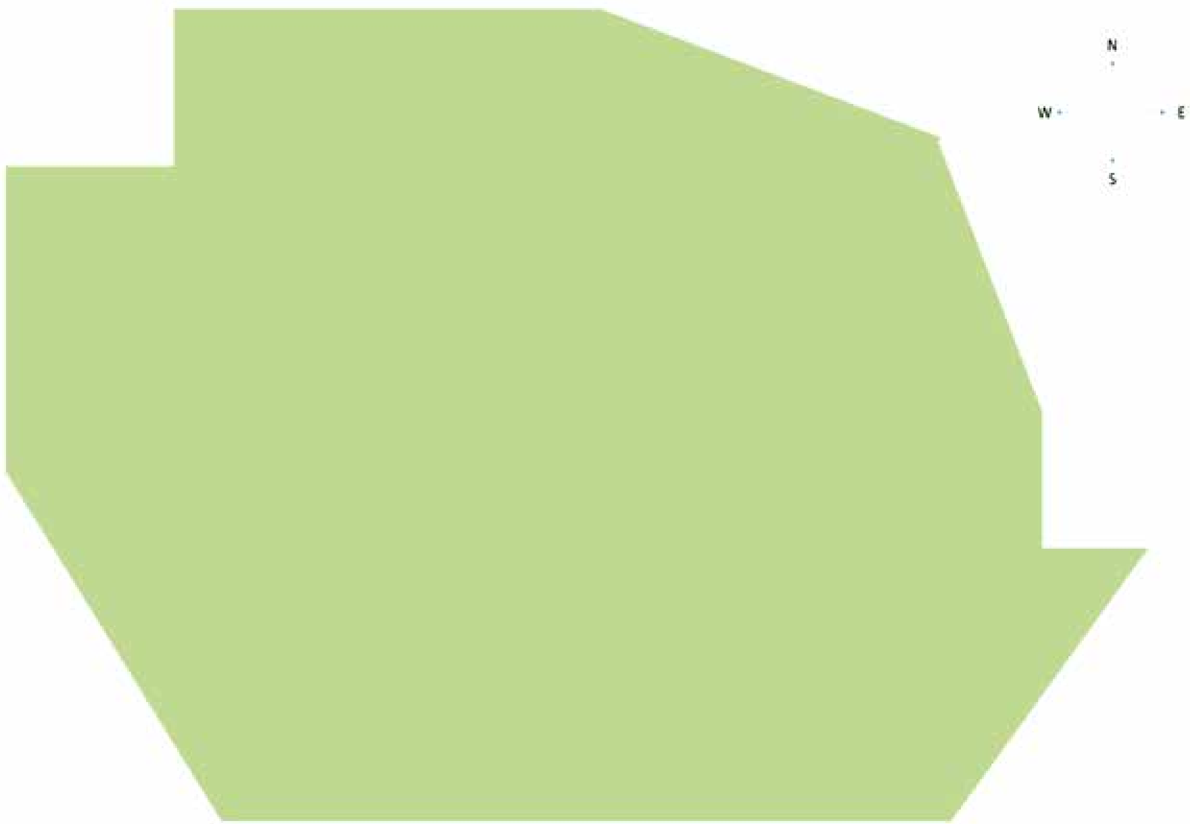
To calculate the clear area needed around each turbine:

1. Consider the two schematics shown above. It shows that:
   a. 80 m must be left clear at every point behind the blades
   b. 40 m must be left clear at every point in front of the blades.
   c. 200 m must be left clear behind the centre of the hub.
   d. 80 m must be left clear around the turbine blades.
2. Consult with each other to design a shape that represents the area required around each turbine. You are welcome to come up with more than one design if you like as long as they satisfy the safety regulations above. There is no one right answer. Be creative!

Task 2: Find the maximum number of turbines possible to place onto the site

Now that you know the dimensions of the shapes, you'll need to figure out how many of each shape you can fit onto the site. Here's how to proceed. For each shape you have designed:

1. Make lots of cut–outs of that shape. Make sure that they are the dimensions that you decided in the last part.
2. The scaling is 1 cm to 100 m.
3. Consult with the engineers in **Group 1** to find out the direction in which the turbine should be pointing.
4. Use this direction to orient the shape on the site (picture given below).
5. Fit as many of the shapes as possible (Be careful though to keep the windmill pointed in the correct way).
6. Report your final answer to the engineers in **Group 5** so they can complete their exercise.



Fill out one project card and hand it in to your project manager.

Congratulations, engineers! You have successfully completed this exercise. If you're done, see if you can assist other teams on your site.

Site 2

Engineer Group 3

Financial Analysts

Engineers, your goal is to analyse the cost and profit from the site. You will need to work with other groups so appoint somebody from your group to be the coordinator.

The project manager, who is also a member of your group, will have the task of representing the whole site. Ensure that your project manager is involved and updated at every stage so that they can make the best case for this wind farm.

Task 1: Cost Find the costs of building a wind farm

There are a lot of costs when it comes to building a wind farm. In fact, we won't even be able to consider all of them but we will consider the most important or obvious ones.

We can think of the costs under the following headings:

1. The Site
2. The Components
3. Maintenance
4. The Engineer

1. The Site:

It costs money to purchase the site. Consult with the land surveyors from Group 2 to get the cost of purchasing the land

It also costs money to prepare the land. This means that you will have to make clear large rocks, big bushes and shrubs and level the land. This will cost a flat rate of €5,000.

What is the total cost of the site including site preparation?

2. The Components:

It will cost to build the wind mills for the site. Each windmill has

a. 1 tower
b. 1 hub
c. 3 blades

The following quotes for the windmill components have been given from 3 different companies Irish Wind, Storm Chasers and Propellers Inc. We are new customers but would hope to become regular customers.

Irish Wind is located in Dublin, roughly 120 km from the wind farm. It has been established and trading in energy systems since 1980.

ITEM	QUOTE FOR NEW CUSTOMERS (€)	QUOTE FOR REGULAR CUSTOMER (€)	EXPECTED LIFETIME (YEARS)
Turbine blade	1,400	1,300	3
Hub	15,000	15,000	6
Tower	20,000	18,000	12

Storm Chasers is located in London, roughly 250 km from the site. It was established in 1995. It is reputed to have supplied many units to several American and British wind energy projects.

ITEM	QUOTE FOR NEW CUSTOMERS (€)	QUOTE FOR REGULAR CUSTOMER (€)	EXPECTED LIFETIME (YEARS)
Turbine blade	1,200	1,000	3
Hub	20,000	18,000	6
Tower	18,000	15,000	12

Propellers Inc. is located in Galway, roughly 50 km from the site. It was established in 2002 and is completely Irish owned.

ITEM	QUOTE FOR NEW CUSTOMERS (€)	QUOTE FOR REGULAR CUSTOMER (€)	EXPECTED LIFETIME (YEARS)
Turbine blade	2,000	1,500	5
Hub	22,000	18,000	10
Tower	21,000	17,000	15

Consider the quotes you have been given for the components. Which company would be most beneficial (either in the short term or in the long term or both) and would maximise the profits? Be prepared to explain your reasoning and communicate your findings with the Project Manager. There are no wrong answers, but there are optimum situations.

How much will the components cost in total?

3. Maintenance:

There are many things that need to be maintained on a wind farm:

- The windmill itself needs to be cleaned regularly to ensure that it is working efficiently.
- If any of the parts fails, it will need to be replaced.
- Even if it does not fail, there will be a recommended lifetime for each component of the windmill. When it has been operating for as long as its recommended lifetime, then it will need to be replaced even if it is still functional.

To calculate the replacement of parts. Consider the lifetime of the components that you decided on and split the cost evenly over the years. For example, if the lifetime of a component is 5 years and costs €5,000 you will assign €1,000 for it a year.

Can you calculate the replacement costs for each of the components from your chosen supplier?

3 Blades:

1 Tower:

1 Hub:

4. The Engineer:

An engineer will be contracted to make regular visits to the wind farm to ensure that all windmills are running as expected. The income they generate from this will be considered their regular salary.

Below there are three candidates. You are provided with some information about their qualifications, their work experience (if they have any) and the salary the candidate would get if they are hired. Discuss among each other and decide who you would hire.

Michael O'Brien

1. Finished his PhD in UCD last year in Wind Energy Systems
2. 5 years working experience in off–shore wind off the coast of Denmark
3. Expected salary: €50,000 per year

Linda McNally

1. Degree in Electrical Engineering with Management
2. Worked as project manager with the Sustainable Energy Authority Ireland on several wind energy projects and has experience in research over the last 10 years
3. Expected salary: €60,000 per year

Jordan Hoffmann

1. Graduated with a Master's degree in alternative energy systems this year
2. No work experience
3. Expected salary: €34,000 per year

Based on the above information, which candidate would you hire and why?

Task 2: Calculate the net profit and prepare your presentation

Consult with the engineers from Group 5 to get the income generated from the wind farm.

Table 1: This table will help you put together all the costs (for one year) that you found from your first task.

ITEM / SERVICE	COSTS
Purchase of site	
Preparation of site	
Cost of components from chosen supplier	
Replacement of parts	
Engineer's salary	
Total costs:	

Comparing the total income generated from the wind farm with the total costs to build the wind farm, can we operate effectively as a business? Explain your reasoning

Congratulations, engineers! You have successfully completed this exercise. If you're done, see if you can assist other teams on your site.

Site 2

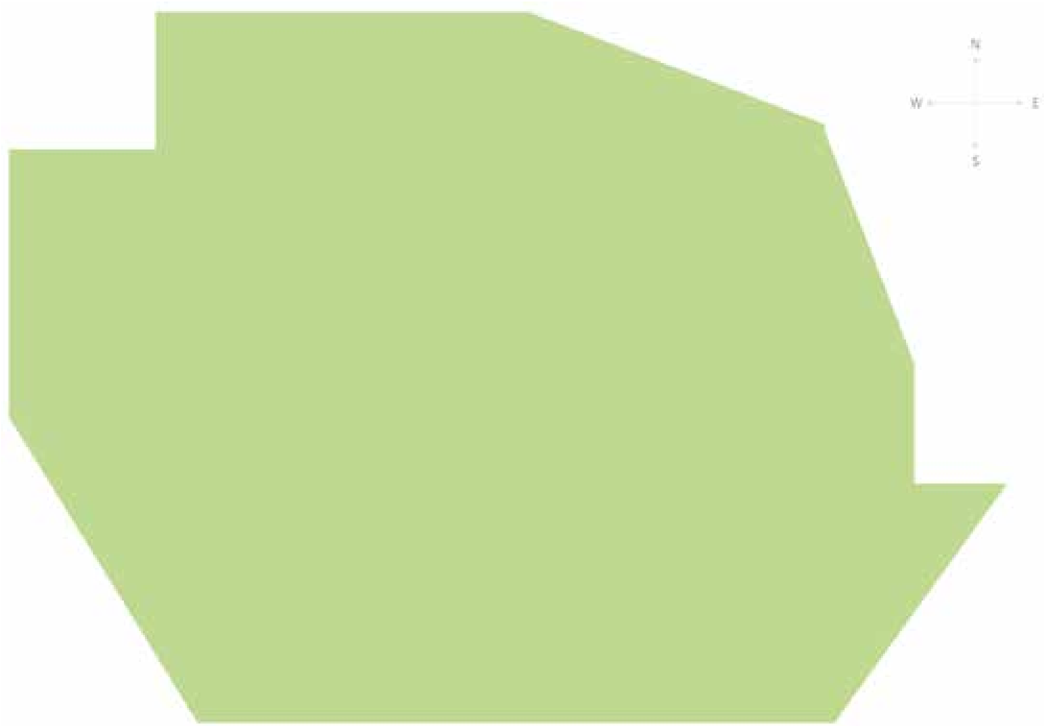
Engineer Group 4

Land Surveyors

Surveyors, your task is to calculate the cost of purchasing the site for the wind farm.
Choose one person who will coordinate with other groups and the project manager. You will need to calculate how big the site is. Then using this information and with the help of the other engineers, you will be able to calculate the total cost of the land. Let's get started!

Task 1: Calculate the surface area of the site.

The shape of the site is given below. The scaling is 1cm = 100 m. This means that every centimetre that your ruler measures, is the same as 0.1 km in the real world.



1. Using a straight edge or ruler, split the above shape into rectangles and triangles.
2. Use Tables 1 and 2 on the next page to calculate the area of each of the shapes, remembering to convert each of your measurement to kilometres.

Table 1: This table will help to calculate the area of the all the rectangles.

RECTANGLE	MEASURE LENGTH, L (in km)	MEASURE WIDTH, W (in km)	AREA: CALCULATE L X W (in km^2)
1.			
2.			
3.			
4.			
5.			
6.			
7			
8.			
9.			
10.			
11.			
12.			
13.			
14.			
15.			
CALCULATE THE TOTAL AREA OF EACH OF THE RECTANGLES:			

Table 2: This table will help you to calculate the area of all the triangles.

TRIANGLE	MEASURE THE LENGTH OF THE BASE, B(in km)	MEASURE PERPENDICULAR WIDTH, W (in km)	AREA: CALCULATE 0.5 X B X W (in km^2)
1.			
2.			
3.			
4.			
5.			
6.			
CALCULATE THE TOTAL AREA OF ALL OF THE TRIANGLES:			

Task 2: Calculate the price of the land

Now that you know how large the area of the land is, you can proceed to calculate the price of the land:

1. The unit price is €30,000 per km^2.
2. Fill in Table 4 to find your final answer.
3. Give your final answers to the engineers in Group 3 so they can complete their challenge.

Table 4: This table will help you to find the total cost of purchasing this piece of land.

SHAPE	PUT IN TOTAL AREAS FROM TABLES 1 AND 2	MULTIPLY BY THE UNIT PRICE
Rectangle		
Triangle		
Totals	Total area =	Total price =

Prepare to report to your Project Manager using the report card

Congratulations, engineers! You have successfully completed this exercise. If you're done, see if you can assist other teams on your site.

Site 2

Engineer Group 5

Electrical Engineers

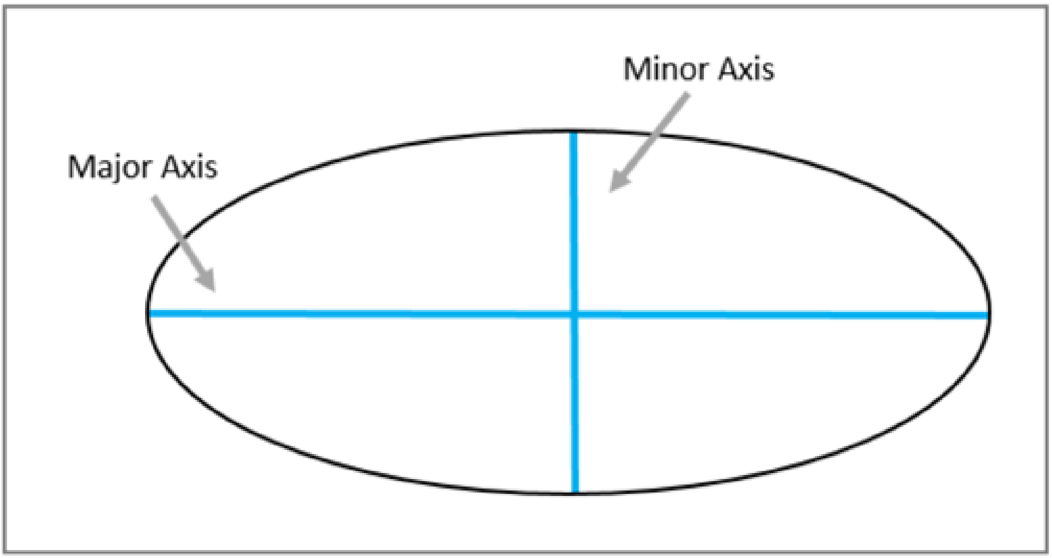
Engineers, your brief is as follows, we need you to work out the maximum wattage the windmills can output and then work with the Financial Analysts (**Group 3**) to calculate our possible profits from the windfarm's energy output.

Your first task is to figure out the maximum wattage the windmills can create in order to tell the safety engineers (**Group 6**).

The way we'll do this is by finding out first how fast the blades will spin when the wind is at its highest. Then find out how many watts this speed will give us.

Task 1: Find the speed of the blades at maximum wind speed

To find the speed of the blades, we first need to find the area of the blades and use what we know about the relationship between the area and the speed of the blades to find their maximum speed. Our first step is to find the area of the blades. The blades of a windmill are essentially shaped like an ellipse (which is the term for a squashed circle or an oval). The information we need to know about an ellipse are the lengths of its minor axes and major axes (which you can see in the diagram below).



The formula for the area of an ellipse is:

$$A = \pi ab$$

Where

A	AREA OF ELLIPSE
a	The length of the semi–minor axis (half the length of the minor axis)
b	The length of the semi–major axis (half the length of the major axis)

Our windmill blades are **4.2m** wide at their widest part and they are **18.8m** long.

Using this can you find out how much area one blade has? How about all three blades?

Now we need to find out how fast the blades go at the maximum wind speed. Because the blades aren't moving in a straight line (they are rotating around the hub), this speed is given in revolutions per hour.

Our mathematicians have found that relationship of:

$$\omega = \frac{A * v * \sin(\theta)}{8.0498}$$

Where

ω	**The angular velocity (speed the blade is turning) of the blades (in revolutions per hour)**
A	**The area of all three blades (in metres squared)**
v	**The speed of the wind (in knots)**
θ	**The angle of the blades (in degrees)**

We've already found A and we know that the blades will be at an angle of 15 degrees, but we still need to know the max speed of the wind which you can get from Group 1, the Windmill Technicians.

Calculate the angular velocity of the blades:

Once you have the maximum angular velocity, all we need is to convert this to a maximum wattage. We know that one revolution per hour will create 0.24 MW (megawatts) of power with these turbines.

Use the figure you found above for angular velocity (ω) to work out the maximum wattage output.

Report your findings to Group 6, the Safety Technicians

Task 2: Calculating the income generated by the wind farm

For this task, you will be working with the Financial Analysts, **Group 3**, to work out how much money the windfarm will make on average over the course of the year.

Firstly, you will need to figure out how much money one windmill makes per year. Then ask **Group 2** how many windmills we will have in order to calculate the windfarm's income per year.

In order to find the amount of money the windmill makes, we need to find out how much energy it will produce.

We know how much power the windmills produce in megawatts from Task 1. We need to convert this to megawatt hours (a unit of energy) by multiplying the power by the amount of time the windmill is producing energy each year.

The windmills will be turned on 12 hours per day, every day of the year, except for Christmas Day, and for one day every 13 weeks for maintenance work.

This is enough information to calculate how many hours the windmill will be on for in a year.

Hours:

Now we just need to find the value, in euro, of that energy. We know that the suppliers of electricity in Ireland buy energy as described in **table 1.**

We will sell energy to each supplier equally (i.e. **25%** of the energy we produce will go to each supplier)

This is enough information to find our annual income, please report to **Group 3** with your findings once you have finished. **Note:** 1 megawatt hour = 1000 kilowatt hour

Table 1: Showing the rate of energy purchased by various suppliers

COMPANY	RATE PURCHASED	PROJECTED AMOUNT FROM EACH COMPANY
Electric Ireland	0.032c per kilowatt hour	
Pinergy	33c per megawatt hour	
Airtricity	0.034c per kilowatt hour	
Energia	€1.5 per 5 megawatt hours	
Annual income total:		

Site 1

Engineer Group 6

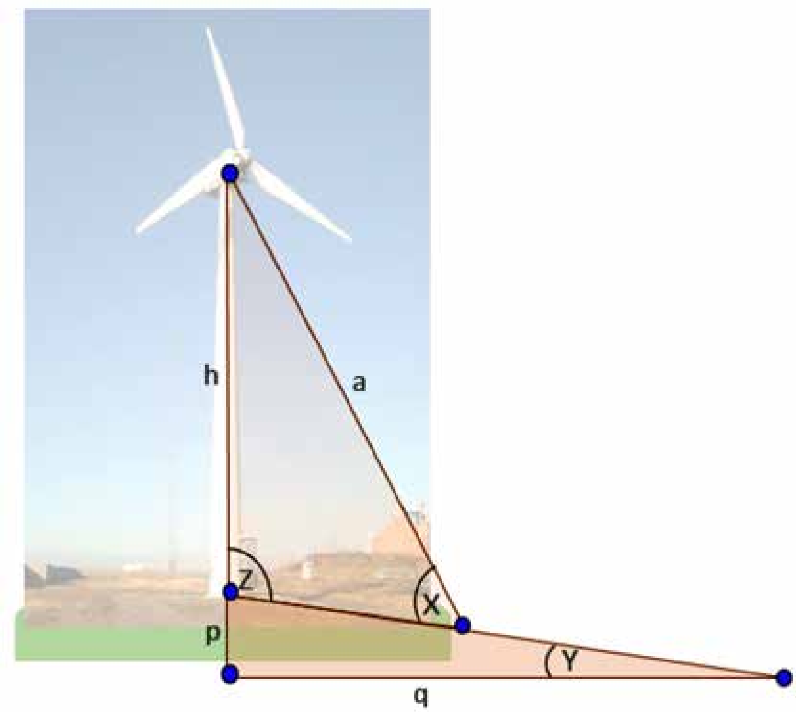
Safety Technicians

Engineers, your brief is as follows, we need you to figure out the maximum safe height of the windmills that are going to be in our energy farm. Given some guidelines about how the windmill has to be anchored to the ground, you will then need to design a safe electric grid to provide energy to the local community.

Your first job is to find the maximum allowable height for the windmills. Under E.U. regulations, the maximum angle **relative to the ground** our anchors are allowed be placed is 65°. Also, the anchors must be attached to the top of the windmill. Lastly, the anchors are limited to being **55m** long.

This is enough information to find the maximum height we can build the windmills on flat ground. However, the windmills will all be placed on the top of artificial hills, where the height drops by **1m** for every **10m** you go out.

What you'll have to do is it put all of this information onto the diagram below:



Where,

h	**Height of windmill**
a	**Length of anchor**
p	**Height of hill**
q	**Width of hill**
x	**Angle of anchor relative to the ground**
y	**Angle of elevation of hill**
z	**Angle between windmill and ground**

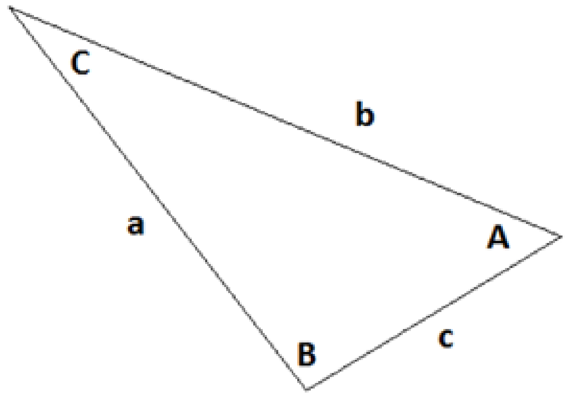
We're looking to find h, but we do this in a number of steps:

1. **Find the slope of the base triangle by finding the rise over the run,** $\frac{p}{q}$ **.**

2. **The slope (m) = tan(Y). Find Y.**

3. **Find the size of the angle z.**

Next we're going to need to use the sine rule, which states that if you have a triangle like the one below:



Then:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

That means that if you knew, say, what a, b and $\sin(B)$ were, you could find $\sin(A)$ using algebra, i.e.

$$\sin(A) = \frac{a * \sin(B)}{\mathrm{b}}$$

So we can now go on to our last steps:

4. Use the sine rule to find the height of the windmill, h.

Once you have found this, report to engineer **Group 1** with your findings.

Task 2: Design the local electric grid

In any electrical system, engineers must be careful of overloads (where an excessive amount of current in the wire causes heat and possible damage to equipment).

For your next task, you are going to have to design the local electric grid to ensure that the likelihood of overloads in the grid is acceptably low, to do this you will first have to find out how many houses you can connect successively for different wattages.

The local council have decided that the probability of an overload during the course of a connections lifetime must be below 0.05 for it to be safe. Our scientists have found that the probability of an overload for n connected houses, with appropriate wattage w is:

$$P\ (overload) = \frac{n}{n + 100 - w}$$

Where,

N	**Number of houses connected**
W	**Wattage (in MW or megawatts)**
p	**Probability of an overload in the connection's lifetime**

Given this, how many houses could we safely connect for these wattages? We've done the first one to get you started.

The first calculation for n = 1 is:

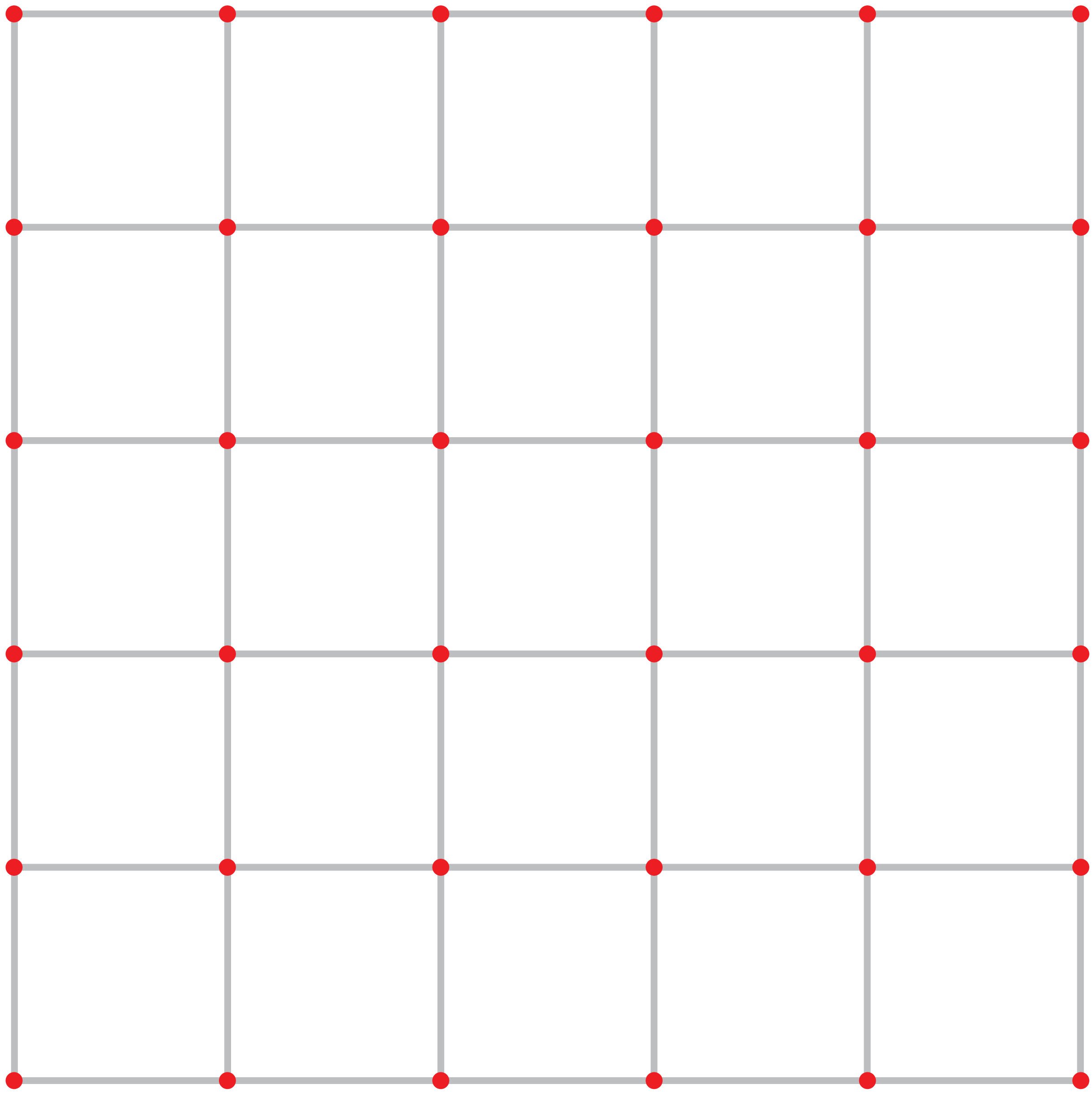
$$\frac{1}{1 + 100 - 10} = 0.011$$

Wattage	Prob. For N=1	Prob. For N=2	Prob. For N=3	Prob. For N=4	Prob. For N=5
10MW	0.011	0.022	0.032	0.043	0.053 (too big)
20MW					
30MW					
40MW					
50MW					

Now ask the Electrical Engineers **(Group 5)** what the maximum wattage the windmills can output and find the corresponding maximum number of houses we can connect.

Then you need to place the smallest number of transformers on this grid so that you can provide energy to all the houses.

Each red dot on the grid represents a house and you can place a transformer on any of the red dots, this provides energy to that house but doesn't use up one of the connections.



Congratulations, engineers! You have successfully completed this exercise. If you're done, see if you can assist other teams on your site.

UCD Student Participants

2015 Maths Sparks

Stephen Alff	Stage 2	BSc. College of Science
Jennifer Costello	Stage 2	BSc. College of Science
Deirdre Creegan	Stage 2	BSc. College of Science
Ciara Delaney	Professional Masters in Education, School of Education	
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Dr Aoibhinn Ní Shúilleabháin
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