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MATHS SPARKS

Problem Solving Workshops

VOLUME II

This resource book is published with the support of SFI Discover, UCD School of Mathematics and Statistics, UCD Access and Lifelong Learning and UCD College of Science.
Published: University College Dublin 2018.

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Contents

Introduction ... 2
Workshop Content ... 4
The Handshake Puzzle ... 5
The Handshake Puzzle: Workshop Outline ... 6
Activity 1 ... 12
Activity 2 ... 14
Activity 3 ... 15
The Birthday Problem ... 17
The Birthday Problem: Workshop Outline ... 18
Activity 1 ... 26
Activity 2 ... 27
Activity 3 ... 29
Numbers, Infinity and Shapes ... 31
Numbers, Infinity and Shapes: Workshop Outline ... 32
Activity 1 ... 40
Activity 2 ... 43
Activity 3 ... 45
Activity 4 ... 46
Activity 5 ... 48
Astronomy Mission ... 49
Astronomy Mission: Workshop Outline ... 50
Astronomy Mission: Formulae Sheet ... 57
Activity 1 ... 58
Activity 2 ... 60
Activity 3 ... 61
Logic ... 63
Logic: Workshop Outline ... 64
Activity 1 ... 75
Activity 2 ... 76
Activity 3 ... 79
UCD Undergraduate Student Volunteers 2017 ... 84
Participating Schools ... 84
Acknowledgements ... 85

Introduction

This is the second volume of the UCD Maths Sparks Problem Solving Workshops booklet.

Maths Sparks is a series of workshops for senior-cycle post-primary students, designed and run by volunteer undergraduate students and academic staff of the UCD School of Mathematics & Statistics. The workshops were created to encourage more young people to consider pursuing mathematics and mathematics-based courses at third level, through demonstration of the applications of mathematics outside of the post-primary school curriculum. Through interactive workshops, based on sense-making and students' articulation of their mathematical thinking (Schoenfeld, 1992; Stein, Engle, Smith & Hughes, 2008), these workshops can provide students with opportunity to explore topics within the broad subject of mathematics and potentially impact their attitudes towards and self-confidence in mathematics (Cronin et al., 2017).

Secondary pupils and undergraduate volunteers for Maths Sparks 2017

We hope this booklet can be used by teachers to encourage their students to investigate and explore mathematical topics. As additional activities, teachers may also like to encourage their students to research the various mathematicians noted throughout the booklet. (These historical figures are hyperlinked to the relevant pages in the web version of this booklet.) Realising that mathematics is a human endeavour and not a static body of knowledge, may 'spark' the interest of students and encourage them to pursue their study of mathematics after their post-primary education.

Further information on Maths Sparks and the first volume of the Maths Sparks workshops are available at:
www.ucd.ie/mathstat/mathsparks/

If you would like to share any feedback or commentary on the workshops, please feel free to contact the editors.

Maths Sparks was supported by SFI Discover, UCD College of Science, UCD Access & Lifelong Learning and UCD School of Mathematics & Statistics.

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Secondary pupils and undergraduate volunteers for Maths Sparks 2017

References

Cronin, A., Ni Shuilleabhain, A., Lewanowski-Breen, E., Kennedy, C. (2017). Maths Sparks: Investigating the impact of outreach on pupil's attitudes towards mathematics. *MSOR Connections*, 15(3), 4-13.

Schoenfeld, A. (1992). Learning to think Mathematically: Problem Solving, Metacognition, And Sense-Making in Mathematics. In Grouws, D (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (334-370). New York: MacMillan.

Stein, M., Engle, R., Smith, M. & Hughes, E. (2008). Orchestrating Productive Mathematical Discussions: Five Practices for Helping Teachers Move Beyond Show and Tell. *Mathematical Thinking and Learning,* 10(4), 313-340.

Workshop Content

The Handshake Puzzle

Introduction

The handshake puzzle is a classic mathematical problem that involves finding the total number of handshakes between finite numbers of people. This puzzle is rooted in an important area of mathematics known as combinatorics, which concerns the study of combinations, permutations, and enumeration of elements within a finite set. It is therefore often considered 'the art of counting'. Whilst aspects of combinatorics are believed to date back to ancient China and India, it was only in the 17th century when the subject significantly began to develop. In fact, much of its development is accredited to well-known mathematicians of the time - **Pascal**, **Fermat** and **Euler** - each of whom discovered important combinatorial results, which contributed to the development of probability theory. More recent advancement in combinatorics is attributed to the growth of computer science and the need for algorithmic methods to solve real-world problems. These have led to applications in a wide range of areas including coding theory, experimental design, and DNA sequencing.

Many fields in mathematics have strong foundations in combinatorics such as graph theory, probability, and statistics. In particular, the "N choose 2" formula, which serves as the basis of the handshake puzzle, is a fundamental concept in probability that can be used to compute a wide range of mathematical problems, as will be demonstrated in this workshop.

Aim of Workshop

The aim of this workshop is to introduce students to the basic concepts in combinatorics, whilst also developing their problem-solving skills through induction and through recognising patterns. Students will be provided with the opportunity to simulate the handshake puzzle in an effort to find a general formula for the problem and also contribute to the development of their team-work and communication skills.

Learning Outcomes

By the end of this workshop students will be able to:

- Explain, in their own words, what is meant by combinatorics
- Calculate the total number of handshakes between any given number of people
- Justify why the general formula for the total number of handshakes is

$$\frac{n(n-1)}{2}$$

- Apply the "N choose 2" formula to various mathematical problems

The Handshake Puzzle: Workshop Outline

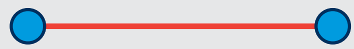
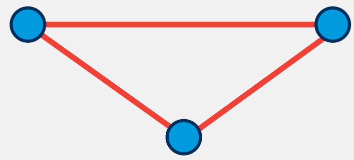
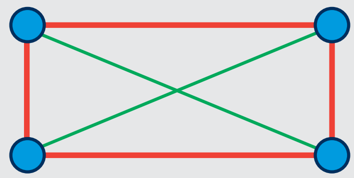
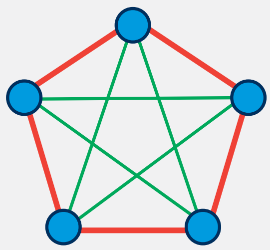
SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
2–5 mins (00:05)	**Introduction to the Handshake Puzzle**	– Introduce the handshake puzzle and outline the rules: 1. Each person must shake hands with every other person in the room 2. Each pair shake hands only once 3. A person cannot shake hands with themselves (Note: Students will be unable to do the activity without these rules)
15 mins (00:20)	**Activity 1** **The Handshake Puzzle**	– **Activity Sheet 1**: Students are asked to determine the number of handshakes between given numbers of people (see **Appendix – Note 1**) – Encourage students to draw diagrams and/or simulate the puzzle in small groups – Facilitate a whole class discussion on students' solutions. – You may wish to use the online simulator as an additional visualisation for students (see link in **Additional Resources**)
10–15 mins (00:35)	**Activity 2** **The Handshake Puzzle**	– **Activity Sheet 2:** Students now attempt to find a general formula for the total number of handshakes (see **Appendix – Note 2**)

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5 mins (00:40)	**Notation**	– Mention that the general formula $$\frac{n(n-1)}{2}$$ is called "N choose 2" – Demonstrate to students that the total number of handshakes for 10 people can be written as $\binom{10}{2}$ or 10C2 (it may be useful to act this out before-hand) – Explain how to compute this on the calculator (press "10", "shift" "nCr" and then "2") – You may wish to ask students what the total number of handshakes would be for a larger group of people (e.g. 120 people)
10 mins (00:50)	**Activity 3** **Pizza Toppings**	– **Activity Sheet 3**: Students are asked to find the total number of unique pairs of pizza toppings (see **Appendix – Note 3**).
5 mins (00:55)	**Combinatorics**	– Mention that the concepts covered in the workshop relate to combinatorics and explain what is meant by this term – Outline some of the applications of combinatorics (see **Workshop Introduction**)

The Handshake Puzzle: Workshop Appendix

Note 1: Solutions for Activity 1

Q1. (i) What is the maximum number of handshakes between the following numbers of people?

NUMBER OF PEOPLE	NUMBER OF HANDSHAKES	DIAGRAM
Two	1	
Three	3	
Four	6	
Five	10	
Six	15	Same idea as above but with 15 lines i.e. each person shakes hands with 5 other people

(ii) Identifying the pattern as quadratic (link to curriculum - Strand 3: Numbers)

NUMBER OF PEOPLE	NUMBER OF HANDSHAKES	DIFFERENCE BETWEEN ROWS: 1ST DIFFERENCE	DIFFERENCE BETWEEN ROWS: 2ND DIFFERENCE
Two	1	+2	+1
Three	3	+3	+1
Four	6	+4	+1
Five	10	+5	
Six	15		

(iii) How many possible handshakes are there between each of the following?

- Nine people = 36 handshakes
- Ten people = 45 handshakes

Note 2: Solutions for Activity 2

Q2. (i) How does Bob's answer compare to your answer in Q1 (iii)?

He multiplied our answer by 2 and therefore double counted the number of handshakes i.e. each pair of people shook hands twice according to Bob's calculations.

(ii) Using Bob's calculations, what could we do to his answer to get the correct number of handshakes in the room?

$$\frac{10(9)}{2} = 45$$

(iii) Would the above method work if Bob was in the room with 8 other people (9 in total)?

Yes:

$$\frac{9(9-1)}{2} = \frac{9(8)}{2} = 36$$

(iv) What if there were 5 other people in the room (6 in total)?

Using the above method, we get

$$\frac{6(6-1)}{2} = 15$$

Compare this with the answer for 6 people in question 1 (i)

(v) Given that n is the number of people in the room, can you find a general formula for the total number of handshakes?

$$\frac{n(n-1)}{2}$$

Explanation: If there are n people in a room, each person will shake hands with the (n - 1) remaining people as they will not be shaking hands with themselves – hence we get (n)(n - 1). However, we need to divide this number by 2 since one handshake allows two people to shake hands.

Note 3: Solutions for Pizza Problem

Q1. (i) How many unique toppings can we have for Topping 1?

4 unique toppings

(ii) Write out all the pairs of toppings that you could have on your pizza:

{Ham, Tomato}	{Ham, Pineapple}	{Ham, Olives}
{Pineapple, Tomato}	{Pineapple, Olives}	~~{Pineapple, Ham}~~
{Tomato, Olives}	~~{Tomato, Ham}~~	~~{Tomato, Pineapple}~~
~~{Olives, Ham}~~	~~{Olives, Pineapple}~~	~~{Olives, Tomato}~~

(iii) How many unique pizzas with only two toppings are there in total?

The order of the toppings does not matter on a pizza, hence there are 6 unique pizzas (crossing out the doubles as above) i.e. {Ham, Pineapple} = {Pineapple, Ham} etc.

Q2. (i) What is the value of n in this activity?

Since there are 4 toppings, the value of n is 4.

(ii) What does each part of the formula $\frac{n(n-1)}{2}$ represent based on the activity we just did?

Total number of unique toppings in our set that we can pick.
In this case n = 4

$$\frac{n(n-1)}{2}$$

For each topping there are n – 1 matching unique toppings.
In this case 4 – 1 = 3

In order to prevent double counting, we halve the total number of pairs e.g. {Ham, Pineapple} = {Pineapple, Ham}

(iii) Using the formula, calculate the number of unique pairs of toppings you can construct from the set {Ham, Pineapple, Tomato, Olives} and check your answer with a calculator.

$$\frac{4(4-1)}{2} = 6$$

To verify your answer on the calculator (instructions for Casio), press "4", then "shift", then "nCr" and then "2".

Sources and Additional Resources:

http://illuminations.nctm.org/Activity.aspx?id=6756 (Handshake puzzle simulator)

http://world.mathigon.org/Combinatorics (Combinatorics)

The Handshake Puzzle: Activity 1

Q1. (i) What is the maximum number of handshakes between the following numbers of people? (Remember the rules of the handshake puzzle.)

(Hint: you may like to draw a diagram and/or simulate the problem in small groups.)

NUMBER OF PEOPLE	NUMBER OF HANDSHAKES	ROUGH WORK/DIAGRAM
Two		
Three		
Four		
Five		
Six		

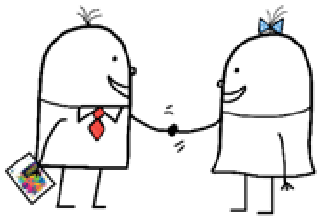
(ii) Can you identify a pattern between the number of people and the number of handshakes?

(Hint: it may be helpful to draw a table.)

(iii) How many possible handshakes are there between each of the following?

Nine people

Ten people



The Handshake Puzzle: Activity 2

Q2. Bob was in a room with 9 other people and he shook hands with each of them exactly once, as did every other person in the room. To find the total number of handshakes, Bob multiplied 10 by 9 and got 90.

(i) How does Bob's answer compare to your answer in Q1 (iii)?

(ii) Using Bob's calculations, what could we do to his answer to get the correct number of handshakes in the room?

(iii) Would the above method work if Bob was in the room with 8 other people (9 in total)?

(iv) What if there were just 5 other people in the room (6 in total)?

(v) Given that n is the number of people in the room, can you find a general formula for the total number of handshakes?

The Handshake Puzzle: Activity 3

You are ordering a pizza and can pick two toppings from the following set:

{Ham, Pineapple, Tomato, Olives}

Q1. We want to find out how many unique pizzas with only two toppings can be made from this set.

Rules:

- The pizza must have two toppings
- The pizzas are chosen by {Topping 1, Topping 2}
- We ***cannot*** use the same topping twice e.g. {Ham, Ham}

(i) How many unique toppings can we have for Topping 1?

(ii) Write out all the pairs of toppings that you could have on your pizza:

(iii) How many **unique** pizzas with only two toppings are there in total?

Q2. Considering our formula from before:

(i) What is the value of n in this activity?

(ii) What does each part of the formula $\frac{n(n-1)}{2}$ represent based on the activity we just did?

Write your answers in the boxes below.

$$\frac{n(n-1)}{2}$$

(iii) Using the formula, calculate the number of unique pairs of toppings you can construct from the set {Ham, Pineapple, Tomato, Olives} and check your answer with a calculator.

The Birthday Problem

Introduction

Probability is a useful mathematical tool that enables us to describe and analyse random phenomena in the world around us. Meteorologists use it to determine the likelihood that it will rain tomorrow, scientists use it to accurately predict the results of their experiments and stock traders use advanced algorithms to calculate the probability of shares falling or rising in value. Whilst many aspects of probability seem relatively intuitive, it often produces some rather unexpected yet remarkable results, as will be demonstrated by the birthday problem below. This problem draws on a range of different techniques and formulae that are central to probability. Here, we try to determine the total number of people needed in a room for there to be a 50% chance that at least two of them share the same birthday (date and month only). Note: It might be useful to do this workshop *after* the "Handshake Puzzle" workshop.

Aim of the Workshop

The aim of this workshop is to introduce students to the basic laws of probability, whilst also developing their ability to analytically deconstruct mathematical questions in order to understand what they are being asked. Additionally, this workshop will show students that probability does not merely concern the act of rolling dice or flipping coins, but rather it is a fascinating and important field of mathematics that can be used to demonstrate some extraordinary results relevant to our everyday lives.

Learning Outcomes

By the end of this workshop students will be able to:

- Explain the pigeonhole principle in their own words and give a contextualised example of the principle
- Apply the complement formula to find the probability of certain events
- Describe, in their own words, what is meant by 'independent events'
- Calculate the probability of two people in their own class sharing the same birthday

The Birthday Problem: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5 mins (00:05)	**Introduction to the Pigeonhole Principle**	– Ask students "If I have 10 red socks and 10 blue socks in a drawer and I take socks out of the drawer randomly without looking, what is the minimum number of socks that I need to take out in order to guarantee a matching pair?" (Answer = 3) – Mention that this is an example of the pigeonhole principle and explain what this means (see **Appendix – Note 1**)
5 mins (00:10)	**Class Activity** **Irish Hair**	– Pose the following question to the class "In Ireland, at least two people have exactly the same number of hairs on their head. How is this true?" (see **Appendix – Note 2**) – Encourage students to discuss the problem in small groups and facilitate a whole class discussion incorporating their suggested solutions
5 mins (00:15)	**Activity 1** **Estimate Sheet**	– **Activity Sheet 1:** Students write down their own birthday and estimates (See **Appendix – Note 3**) – (Note: this activity sheet should be cut in half in advance of the lesson to reduce use of paper) – Collect this activity once students have completed this task, as these will be used later to see if there is a match

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5–10 mins (00:25)	**Revision of probability**	– You may wish to revise probabilities, independent events, and the complement formula with students (see **Appendix – Note 4**)
10–15 mins (00:40)	**Activity 2** **Coin Tosses**	– **Activity Sheet 2:** Students answer questions related to coin tosses (See **Appendix – Note 5**)
15 mins (00:55)	**Activity 3** **The Birthday Problem**	– **Activity Sheet 3:** In pairs, students attempt to solve the birthday problem (see **Appendix – Note 6**) – If students are stuck, encourage them to look over the previous activity
5 mins (01:00)	**Class Match**	– Looking through the students' birthdays on Activity 1, see if there is a match in the class – Discuss the estimates on Activity Sheet 1 – You may like to ask students what the probability would be for their own class size

The Birthday Problem: Workshop Appendix

Note 1: The Pigeonhole Principle

The pigeonhole principle states that if you have n items and are sorting them into m categories, where m is less than n, then at least one of the categories must have more than one item in it. For example, if there are 5 people living in your house, but you only have 4 bedrooms, then at least 2 people must share a bedroom. Whilst this principle may seem intuitive, it can be used to show some rather interesting results - for example, that there are at least two people in Ireland with the same number of hairs on their head or that there are two books in a library with the exact same number of pages. The first formalisation of the pigeonhole principle is accredited to **Peter Gustav Lejeune Dirichlet** and is therefore also commonly referred to as 'Dirichlet's box principle'. Whilst it is widely used in practical problems relating to probability and statistics, it also has applications in computer science, combinatorics and economics.

Note 2: Solution for Irish Hair Activity

The Irish Hair puzzle is a well-known illustration of the pigeonhole principle. The population of Ireland is approximately 4.6 million - however, the average number of hairs on the human head is only 150,000. By the pigeonhole principle, there must be at least two people in Ireland with the same number of hairs on their head. In other words, even if there were 150,000 people, each with a different number of hairs on their head, the rest of the population must fit into one of the categories between 0 and 150,000 hairs.

Note 3: Solution for Activity 1

By the pigeonhole principle, you would need to have 366 people in a room in order to have a 100% chance (a guarantee) that at least 2 people share the same birthday (Note: for this workshop, we are assuming a 365-day year. However, if using the leap year model, just add one to the number of days).

Note 4: Probability Revision

The probability of an event happening:

$$\mathbb{P}\,(\textit{Event happening}) = \frac{\textit{number of successful outcomes}}{\textit{total number of possible outcomes}}$$

Example questions to ask students:

Q1. What is the probability of getting an even number on a fair 6-sided die?

$$\mathbb{P}\,(\textit{Rolling an even number}) = \frac{3}{6} = \frac{1}{2}$$

Q2. What is the probability of getting an ace in a standard 52-card pack?

$$\mathbb{P}\ (\textit{Getting an ace}) = \frac{4}{52} = \frac{1}{13}$$

Independent Events

Two events, A and B, are said to be independent if the probability of A does not affect the probability of B occurring - otherwise, they are dependent. Examples of independent events include flipping a coin or rolling a die.

To calculate the probability of two or more independent events, we multiply the probabilities of the individual events.

$$\mathbb{P}(\text{A and B}) = \mathbb{P}(\text{A}) \times \mathbb{P}(\text{B})$$

$$\mathbb{P}(\text{A and B and C}) = \mathbb{P}(\text{A}) \times \mathbb{P}(\text{B}) \times \mathbb{P}(\text{C})$$

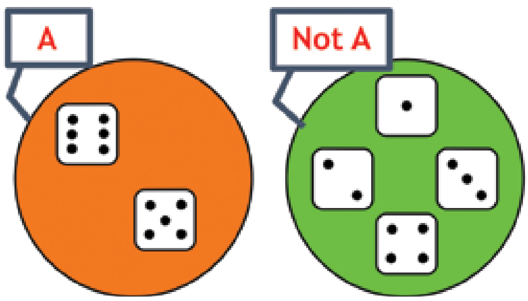
Example questions to ask students:

Q1. You roll a fair 6-sided die five times and you land on 3 each time. What is the probability that the next roll will also give you a 3?

Whilst some may believe that the die is overdue to land on a number other than 3, the probability of getting a 3 is still 1/6 since the events are independent i.e. rolling the die five times will not affect the probability of the next roll.

The Complement:

The complement of an event A occurring, is the event that A does not occur.



In the example above, the event we are interested in is called event A and the complement is referred to as **Not A**. We see that $\mathbb{P}(\text{A}) + \mathbb{P}(\text{Not A}) = 1$ which implies that $\mathbb{P}(\text{A}) = 1 - \mathbb{P}(\text{Not A})$.

Example questions to ask students:

Q1. When the event is drawing a heart from a deck of cards, what is the complement?

Drawing a diamond, club or spade from the deck i.e. **NOT** drawing a heart.

Q2. What is the probability of not getting a 3 on a fair 6-sided die?

$$\mathbb{P}\ (\textit{Getting a 3}) = \frac{1}{6}$$

$$\mathbb{P}\ (\textit{Not getting 3}) = 1 - \mathbb{P}\ (\textit{Getting a 3})$$

$$\mathbb{P}\ (\textit{Not getting 3}) = 1 - \frac{1}{6} = \frac{6}{6} - \frac{1}{6} = \frac{5}{6}$$

Note 5: Solutions for Activity 2

Q1. What is the probability of landing on tails only once in 3-coin tosses?

The 8 possible outcomes are: {TTT, HTT, THT, TTH, HHT, THH, HTH, HHH}

There are 3 outcomes with only one tail: {HHT, HTH, THH}

Hence $\mathbb{P}$(Landing on tails only once) = $\frac{3}{8}$

Q2. What is the probability of NOT landing on tails in 1-coin toss?

$\mathbb{P}$ (Not landing on tails) = $\frac{1}{2}$

This is the same as the probability of landing on heads

Q3. What is the probability of NOT landing on tails in 2-coin tosses?

Since they are independent events, we use the formula: $\mathbb{P}$(A and B) = $\mathbb{P}$(A) $\mathbb{P}$(B)

Hence, $\mathbb{P}$(Not landing on tails) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Q4. What is the probability of NOT landing on tails at all in 3-coin tosses?

$\mathbb{P}$(*Not landing on tails*) = $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ or $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

Q5. In your own words, what would be the complement of the event in Q4?

Landing on tails *at least* once in 3-coin tosses

Note: Anticipate that some students may say "Landing on tails all three times"

Q6. What is the probability of landing on tails AT LEAST once in 3-coin tosses?

It would be quite tedious to calculate this probability directly since "at least once" implies that we could get 1, 2 or 3 tails. Instead, we can use the complement formula since it is easier to calculate the probability of not landing on tails at all in 3-coin tosses

$$\mathbb{P}(\textit{At least}\text{ one tails}) = 1 - \mathbb{P}\text{ (No tails)}$$

$$\mathbb{P}(\textit{At least}\text{ one tails}) = 1 - \left(\frac{1}{2}\right)^3 \text{ (from Q4)}$$

$$\mathbb{P}(\textit{At least}\text{ one tails}) = 1 - \frac{1}{8} = \frac{7}{8}$$

Note: Be sure to ask students how they got $\frac{1}{8}$ as they may skip the middle step

Q7. What is the probability of landing on tails AT LEAST once in 4-coin tosses?

$$\mathbb{P}(\textit{At least}\text{ one tails}) = 1 - \mathbb{P}\text{ (No tails)}$$

$$\mathbb{P}(\textit{At least}\text{ one tails}) = 1 - \left(\frac{1}{2}\right)^4$$

$$\mathbb{P}(\textit{At least}\text{ one tails}) = 1 - \frac{1}{16} = \frac{15}{16}$$

Q8. What is the probability of landing on tails AT LEAST once in 10-coin tosses?

$$\mathbb{P}(\textit{At least}\text{ one tails}) = 1 - \mathbb{P}\text{ (No tails)}$$

$$\mathbb{P}(\textit{At least}\text{ one tails}) = 1 - \left(\frac{1}{2}\right)^{10}$$

$$\mathbb{P}(\textit{At least}\text{ one tails}) = 1 - \frac{1}{1024} = \frac{1023}{1024}$$

Note 6: Solutions for Activity 3

Q1. What is the probability of 2 people sharing the same birthday?

The first person can have any birthday i.e. they have 365 options so the probability that they will have any birthday is $\frac{365}{365}$.

If the second person is to have the same birthday, they only have one option for their birthday, so the probability is $\frac{1}{365}$

Hence, $\mathbb{P}(\textit{2 people sharing the same birthday}) = \frac{365}{365} \times \frac{1}{365} = \frac{1}{365}$

Q2. What is the probability of 2 people NOT sharing the same birthday?

$\mathbb{P}(\textit{2 people sharing the same birthday}) = \frac{1}{365}$
$\Rightarrow \mathbb{P}(\textit{2 people not sharing the same birthday}) = 1 - \frac{1}{365} = \frac{364}{365}$

Alternatively, we know that the first person has 365 options for their birthday and the second person will therefore have 364 remaining options.
$\Rightarrow \mathbb{P}\ (\textit{2 people not sharing the same birthday}) = \frac{365}{365} \times \frac{364}{365} = \frac{364}{365}$

Q3. What is the probability of 3 people NOT sharing the same birthday?

When we have 3 people, we are comparing 3C2 = $\binom{3}{2} = \frac{3(2)}{2}$ = 3 different pairs of people.

From the previous question, we know that the probability of 2 people not sharing a birthday is $\frac{364}{365}$. Since we have **3** pairs, we get $\left(\frac{364}{365}\right)^3$ as they are independent events.

Hence $\mathbb{P}(\textit{3 people not sharing a birthday}) = \left(\frac{364}{365}\right)^3$ = 0.9918 or 99.18%

Q4. How many possible pairs of people can we have in a group of 23 people?

(23 choose 2) = 23C2 = $\binom{23}{2} = \frac{23(22)}{2}$ = 253 possible pairs

(Similar idea to the total number of handshakes between 23 people)

Q5. What is the probability that AT LEAST 2 people out of 23 share a birthday?

Since it is easier to calculate the probability that nobody shares a birthday, we can use the complement formula.

$\mathbb{P}$*(At least two people share a birthday)* = 1 – $\mathbb{P}$*(Nobody shares a birthday)*

We can simplify this to $\mathbb{P} = 1 - (X)^{Y}$ where

X = the probability that, in one pair, they do not share a birthday

Y = the number of times you compare 2 people i.e. how many possible pairs

From Q2 we know $X = \frac{364}{365}$ and from Q3 we know Y = 253

Therefore, $\mathbb{P}$*(At least two people share a birthday)* $= 1 - \left(\frac{364}{365}\right)^{253}$ = 0.5005 or just over 50%

The birthday problem demonstrates that even in a group of only 23 individuals, there is a 50% chance that at least 2 share the same birthday. Whilst this may seem rather remarkable, it intuitively makes sense when considering that we are not just comparing one person to the rest of the group, but rather, we are drawing comparisons between every possible pair of individuals – 253 pairs!

Alternatively, (perhaps for students in senior cycle who have already met factorials) we can find the probability that nobody shares a birthday by considering that the first person has 365 options for their birthday, the second person has 364 options and so on. We therefore get:

$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365}$ ... and so on.

The general formula for this method requires the use of factorials:

$\mathbb{P}$*(At least two people share a birthday)* $= 1 - \frac{365!}{365^{23}((365\text{-}23)!)}$

However, despite the fact that your calculator has more power than the computers that guided the Apollo 11 mission, it is still not powerful enough to compute the above expression. We must therefore use an online calculator in order to get our answer (see link in **Additional Resources**).

That being said, we could also use the following formula (which your calculator can actually manage!):

$\mathbb{P}$*(At least two people share a birthday)* $= 1 - \frac{365P23}{365^{23}}$

Sources and Additional Resources

https://www.ucd.ie/t4cms/Pigeonhole_principle.pdf (Pigeonhole principle)

https://mindyourdecisions.com/blog/2008/11/25/16-fun-applications-of-the-pigeonhole-principle/ (Additional applications of the pigeonhole principle)

https://www.scientificamerican.com/article/bring-science-home-probability-birthday-paradox/ (Birthday problem)

https://www.symbolab.com/solver (Online calculator)

The Birthday Problem: Activity 1

Q1. How many people would you need to have in a room in order to guarantee (i.e. have a 100% chance) that at least 2 people share the same birthday?

You would need to have _______________ people in a room in order to have a 100% chance that ***at least 2 people*** share the same birthday.

Q2. How many people do you think you would need to have in a room in order to have a 50% chance that at least 2 people share the same birthday? Please fill in your guess below:

You would need to have _______________ people in a room in order to have a 50% chance that ***at least 2 people*** share the same birthday.

NAME	BIRTHDAY

Now write down your name and birthday (Day and month only)

The Birthday Problem: Activity 1

Q1. How many people would you need to have in a room in order to guarantee (i.e. have a 100% chance) that at least 2 people share the same birthday?

You would need to have _______________ people in a room in order to have a 100% chance that ***at least 2 people*** share the same birthday.

Q2. How many people do you think you would need to have in a room in order to have a 50% chance that at least 2 people share the same birthday? Please fill in your guess below:

You would need to have _______________ people in a room in order to have a 50% chance that ***at least 2 people*** share the same birthday.

NAME	BIRTHDAY

Now write down your name and birthday (Day and month only)

The Birthday Problem: Activity 2

Q1. What is the probability of landing on tails only once in 3-coin tosses?
(Hint: List all the possible outcomes {e.g. HHH, HHT ... })

Q2. What is the probability of NOT landing on tails in 1-coin toss?

Q3. What is the probability of NOT landing on tails in 2-coin tosses (i.e. not getting tails on the first toss AND not getting tails on the second toss)?

Q4. What is the probability of NOT landing on tails at all in 3-coin tosses? Looking at your list in Q1, how does this answer make sense?

Q5. In your own words, what would be the complement of the event outlined in Q4?

Q6. What is the probability of landing on tails AT LEAST once in 3-coin tosses?

(Hint: Think about the complement formula!)

Q7. What is the probability of landing on tails AT LEAST once in 4-coin tosses?

Q8. What is the probability of landing on tails AT LEAST once in 10-coin tosses?

The Birthday Problem: Activity 3

For each of the following questions we assume: that all birthdays are independent, that a person has an equal chance of being born on any day, and that there are 365 days in the year.

Q1. What is the probability of 2 people sharing the same birthday? (Hint: think about the number of options each person will have for their birthday)

Q2. What is the probability of 2 people NOT sharing the same birthday?

Q3. What is the probability of 3 people NOT sharing the same birthday? (Hint: Think about independent events and the number of pairs you are comparing when you have 3 people!)

Q4. **Suppose there are 23 people in your class. How many possible pairs of people can we have in a group of 23?**

Q5. **What is the probability that AT LEAST 2 people out of 23 people share a birthday?**
(Hint: think about the complement!)

Numbers, Infinity and Shapes

Introduction

Geometry is, perhaps, one of the most practical aspects of mathematics given its relevance to the world in which we live. Whilst it began as a need to measure shapes and define the relations between them, it now has important applications in a wide range of disciplines including navigation, construction, graphic design, fashion design, and robotics. Different mathematical numbers form the basis of many concepts in geometry, with none more fundamental than π (pi) and its myriad of properties. However, the true meaning of π is often hidden behind a button on a calculator and many do not realise that it is actually the ratio of a circle's circumference to its diameter.

A topic that is closely associated with geometry is topology; the study of relations between shapes and spaces. In fact, the Möbius strip is perhaps one of the most famous topological shapes and is unusual in that it is one-dimensional yet exists in a three-dimensional world.

Aim of workshop

The aim of this workshop is to instil a newfound appreciation for the number π and its wide range of applications and properties. In particular, this workshop hopes to encourage experiential learning of mathematics as students actively work together to approximate the value of π using only common everyday materials. The 'Buffon's Needles' trial will similarly be carried out to show students that π can be found in the most unexpected of places. Additionally, this workshop aims to introduce the topic of topology through the construction of a Möbius strip and exploration of its properties.

Learning Outcomes

By the end of this workshop students will be able to:

- Name various mathematical constants
- Derive the relationship between the diameter and circumference of a circle
- Recognise and apply π as a ratio
- Identify and construct a shape with only one face and one edge

Materials and Resources

Party hats (or any other circular based object), string, rulers/metre sticks, scissors, paper, Sellotape, 100 assorted wooden sticks (the size of matchsticks) per group

Key Words

Topology
Topology is the mathematical study of geometrical properties and spatial relations that are preserved through the deformation of objects

Möbius strip
A Möbius strip is a surface with only one face and one edge, formed by rotating one end of a rectangular strip of paper 180° about the longitudinal axis and attaching this end to the other to form a continuous loop

Numbers, Infinity and Shapes: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5 mins (00:05)	**Introduction to Famous Numbers**	– Ask students to write down as many important/famous numbers as they can. – Discuss answers as a class (see **Appendix – Note 1**). – Highlight the importance of π in geometry.
5 mins (00:10)	**Geometry Discussion**	– You may wish to ask students what they know about the history of geometry or its applications. – Provide students with a brief background on the topic (see **Appendix – Note 2**).
15 mins (00:25)	**Activity 1 Calculating π**	– Divide students into groups of 3 and ask them to think of a team name using either a famous number or mathematician e.g. Apple πs, Terrific Trapezoid, the Quaternions etc. – Provide each group with a party hat (or substitute object), ruler & string. – **Activity Sheet 1**: In their groups, students attempt to calculate π using only the above materials (i.e. students can use the string to find a value for the circumference of the circle, radius of the circle etc.) – Once everyone has finished, write down each group's value for π on the board. (Note: A pre-prepared spreadsheet projected onto the board may help with this). Which team got the most accurate value for π? What is the average for the class? – Ask students what other circular objects they used and what values they got for π.

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
15 mins (00:40)	**Activity 2** **Buffon's Needles**	– Hand out 100 wooden sticks and the Buffon's Needles template sheet to each group and explain the task. – **Activity Sheet 2**: In their groups, students try to approximate π using the wooden sticks. Students should throw the wooden sticks randomly on the piece of paper and then fill in the summary table on the activity sheet (Questions you might ask during the activity and further details are available in **Appendix – Note 3**). – Write down each group's value for π on the board. What is the average for the class? Which team got the most accurate value for π? – Mention the work of the Chudnovsky Brothers (see **Appendix – Note 4**).
10 mins (00:50)	**Activity 3** **Shapes**	– Ask the class "What is a shape?", "How do we categorise shapes?" (By the number of faces and edges they have). – Hand out scissors, paper and Sellotape. – **Activity Sheet 3:** Students are asked to draw and construct shapes with a specified number of faces and edges (see **Appendix – Note 5**).
5 mins (00:55)	**Constructing the Möbius Strip**	– Demonstrate how to construct a Möbius strip (see video link in **Additional Resources**) and ask students to make their own (students can now attempt the rest of Activity 3 if they haven't already done so).

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
10 mins (01:05)	**Activity 4** **Möbius Strip**	– **Activity 4:** Students try to predict what will happen to their Möbius strip when it is cut (see **Appendix – Note 6**).
15 mins (01:20)	**Activity 5** **Möbius Strip (Optional)**	– Hand out extra pieces of paper. – **Activity 5**: Students make strips with varying numbers of half loops and attempt to identify a pattern (see **Appendix – Note 7**).
5 mins (01:25)	**Klein Bottle**	– Ask students "What do you think would happen if you stick two Möbius strips along their edges?" – Describe the Klein Bottle (see **Appendix – Note 8**). – You may wish to show students the Klein Bottle video clip (see link in **Additional Resources**).
5 mins (01:30)	**The Doughnut and the Coffee Cup**	– Describe what is meant by topology and ask students how a coffee cup may be considered the same (topologically) as a doughnut before explaining it to the class (see **Appendix – Note 9**). – Ask students if a doughnut and a pretzel would be the same from a topological point of view.

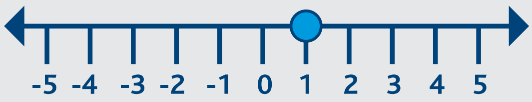
Numbers, Infinity and Shapes – Workshop Appendix

Note 1: Important and Famous Numbers

Below are some of the most important and famous numbers in mathematics

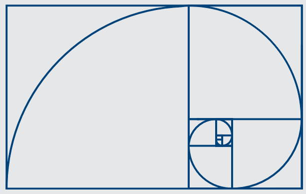
Mathematical Constant

1



The Golden Ratio

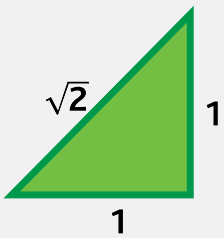
$\varphi \approx 1.6180...$



Pythagoras' constant

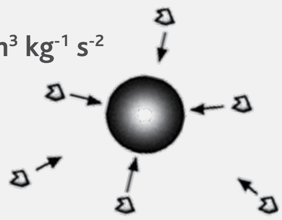
$\sqrt{2} \approx 1.41421356...$

$1^2 + 1^2 = \sqrt{2}^2$



Newton's Gravitational Constant

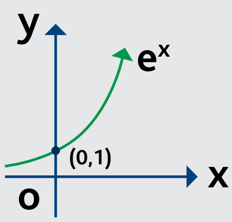
$G \approx 6.67408 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$



Euler's number e

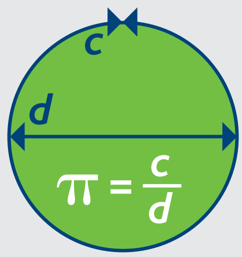
$e \approx 2.71828...$

$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$



Archimedes' constant

$\pi \approx 3.14159...$



Note 2: Geometry Discussion

For the class discussion on Geometry, you may wish to mention some of the following points:

- The word geometry stems from the Greek word 'Geo' meaning earth and 'metria' meaning measure - therefore, it is the 'measure of the earth'
- Eratosthenes (276 - 194 BC) was chief librarian at the Great Library of Alexandria in Egypt and made an amazing approximation of the circumference of the Earth using a few measurements and simple, yet brilliant, geometry
- Euclid had, perhaps, the most profound impact on the development of geometry and his book 'The Elements' has become one of the most important and influential texts in the history of mathematics
- There are many different types of geometry including Euclidean, non-Euclidean, differential etc.

Geometry is a very practical area that has many important applications in today's world including molecular modelling, architecture, engineering, biology, fashion design and graphic design (this list is by no means exhaustive!)

Note 3: Solutions for Activity 2

The Buffon's Needle experiment is an interesting and unusual method for approximating the value of π without the need for circles or spheres. It involves dropping needles on a sheet of lined paper and calculating the proportion of needles that intersect with a line. This experiment was first posed by **Georges-Louis Leclerc,** Comte de Buffon, in the 18th century and is therefore named 'Buffon's Needles' in his honour.

Figure 1: Post-primary students working on the Buffon's Needles activity during a UCD Maths Sparks workshop in 2017

Q1. In your groups, throw the wooden sticks randomly on the piece of paper and then fill in the summary table below.

Students may wish to throw the wooden sticks in groups of 20 or 50 rather than all at once. Note: For this activity, it is important that the length of the wooden sticks be approximately half the distance between the lines on the template (see Figure 1 for example).

Q2. Using this formula, what value did you get for π?

Expect values between 1 and 5

Q3. How accurate is your value for π?

Values between 2.9 and 3.3 would be very good!

Q4. How could you make your value for π more accurate? Any suggestions?

- Take several measurements and calculate the average
- Use a range of different sized circles and calculate the average

Q5. Could we ever measure an exact value for π? Explain your reasoning.

No, it is impossible! π is an irrational number and cannot be expressed as a fraction – it has no ending digit, nor does it contain a repeating decimal pattern!

Note 4: David and Gregory Chudnovsky

Mathematicians and brothers, **David and Gregory Chudnovsky,** are renowned for building a supercomputer (known as m-zero) in their tiny apartment in Manhattan in an effort to explore the properties of π. Whilst no evident pattern emerged from their algorithm, they did succeed in calculating π to approximately two billion places; a world record at the time. Remarkably, π has now been calculated to 22 trillion digits and if you were to print out this file, it would be equivalent to several million books, each containing a thousand pages!

Note 5: Solutions for Activity 3

Q1. Cut out the following two shapes (you may wish to sketch them first!)

(i) A shape with 2 faces and 4 edges

Square, rectangle, rhombus or any irregular 4-sided shape.

(ii) A shape with 2 faces and 1 edge

Circle or oval.

Q2. Can you construct a shape with 1 face and 1 edge using only paper, Sellotape and scissors? Can you prove that your shape only has 1 face and 1 edge using only a pencil?

Allow students to attempt this activity before showing them how to construct a Möbius strip (see video link in **Additional Resources**).

To prove that the Möbius strip has only one face, you can trace a line down its centre until you reach the starting point again (see dotted line in Figure 2). Close inspection will show that this line traversed the entire strip, thereby proving that it only has one face.

Figure 2: Showing that a Möbius strip only has one face and one edge

Similarly, to show that the Möbius strip only has one edge, trace out the edge with a marker and continue around the shape until you reach the starting point again (see Figure 2).

Note 6: Solutions for Activity 4

The Möbius Strip:

Q1. Cut your strip in two, lengthwise along the centre. What do you predict will happen?

You get one long twisted loop with two faces.

Q2. Make a new Möbius strip but this time, try cutting along one-third of its width. What do you predict will happen?

You get two interconnected loops with one larger than the other.

Q3. Make a strip with one full twist and cut it in two, lengthwise along the centre. What do you predict will happen?

You get two full twist loops that are tangled together.

Figure 3: Maths Sparks students and volunteers demonstrating that a Möbius strip only has one face

The Story of Niamh:

Q. What is Niamh's mistake? Please explain your reasoning.

Niamh has miscalculated, but you might enjoy the discussion around this with your students. When we mark a start point and measure the distance walked by Niamh, she walked two kilometres. However, without measuring this specific distance, there is an infinite walkway. Have **fun!**

Note 7: Solutions for Activity 5

Q1. Try making some strips with 2, 3, 4 or 5 half twists (i.e. 180° rotations). How many faces and edges does each one have? Cut these new strips in half and describe what happens!

NUMBER OF HALF TWISTS	NUMBER OF FACES	NUMBER OF EDGES	WHEN CUT IN HALF
1	1	1	1 large loop - 2 faces
2	2	2	2 interconnected loops - 2 faces
3	1	1	1 twisted large loop - 2 faces
4	2	2	2 interconnected loops - 2 faces
5	1	1	1 very twisted large loop - 2 faces

Q2. Do you notice any pattern?

Where there is an odd number of half twists, the resulting shape only has 1 face and 1 edge. When the number of half twists is even, the resulting shape has 2 faces and 2 edges.

Note 8: The Klein bottle

The Klein bottle is a closed, one-sided topological surface that was first proposed by the German mathematician, **Felix Klein**, in 1882 (see Figure 4). It can be formed by gluing two Möbius strips together along their edges, as demonstrated by the video clip included in the **Additional Resources** section. If we were to trace along the surface of the Klein bottle, we would go from what we might perceive as being the 'inside' of the bottle to the 'outside' without actually crossing the boundary between the two. Consequently, we cannot strictly define an inside or outside for the Klein bottle. Nevertheless, the examples of Klein bottles that are commonly depicted in textbooks or on the internet are actually three-dimensional representations of what is, in fact, a four-dimensional surface. A true Klein bottle, for example, does not actually intersect with itself yet this property is impossible to replicate in the three-dimensional models. It is therefore difficult for us to visualise what a true Klein bottle might look like.

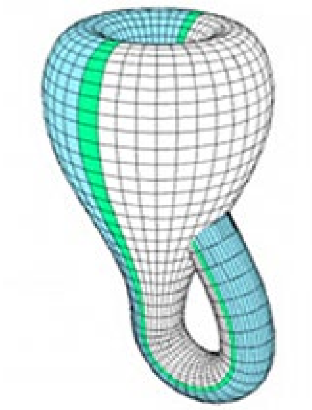


Figure 4: Klein bottle

Figure 5: Doughnut and the Coffee cup.
Image Credit: https://www.shapeways.com

Note 9: The Doughnut and the Coffee Cup

In topology, we look for properties of objects that do not change when they are deformed continuously i.e. without breaking, tearing or gluing. A coffee cup and a doughnut are therefore considered the same from a topological point of view, since both have one hole going all the way through and this property does not change when one is turned into the other (see Figure 5). However, a doughnut is not considered the same as a pretzel since they each have a different number of holes!

Sources and Additional Resources

https://www.youtube.com/watch?v=Cce1XJ3BCxg (Eratosthenes' calculation of the circumference of the Earth)
https://www.geogebra.org/m/zqwhWcfS (Buffon's Needles simulator)
http://www.newyorker.com/magazine/1992/03/02/the-mountains-of-pi (Chudnovsky brothers)
https://brilliant.org/wiki/mobius-strips/ (The Möbius strip)
https://www.youtube.com/watch?v=Z30c5wvoS_s (Constructing a Möbius strip video)
https://plus.maths.org/content/imaging-maths-inside-klein-bottle (Klein bottle)
https://www.youtube.com/watch?v=E8rifKlq5hc (Klein bottle video)

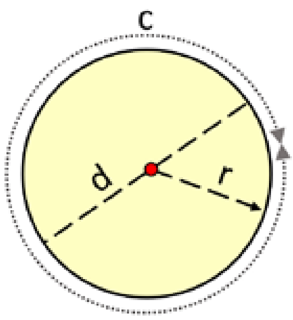
Numbers, Infinity and Shapes: Activity 1

Team Name:

Let's forget about π (pi) and pretend that **Archimedes** never found a value for it. As part of his mathematical team, what might you need to calculate it?

- A metre stick/ruler
- A long piece of string
- A circular object
- A calculator

Task One: Reformulate!



We hypothesise that a relationship exists between the circumference of the circle and the diameter.

c = circumference **d = diameter** **r = radius**

Our hypothesis is that for **any circle**, the circumference is equal to some number times the diameter. Let's call this number banana.

Q1 We therefore say that c = banana x d. Rearrange to find banana in terms of c and d

Q2. Repeat, but this time, let d = 2r (i.e. the diameter is twice the radius)

Q3. **You should all have a party hat. Let's assume its base is perfectly circular (it should be pretty close!). We want you to calculate the value of banana for this party hat using only a metre stick/ruler, string, your calculator and most importantly, your common sense!**

You can use the box below for rough work and then fill your answer into the table.

Once you have finished, you may wish to calculate banana for other circular objects, which you can also fill into the table.

OBJECT	VALUE FOR C	VALUE FOR R	VALUE FOR BANANA

Numbers, Infinity and Shapes: Activity 2 Template

Numbers, Infinity and Shapes: Activity 2

Let's move onto something completely random - literally, random! You should have around 100 assorted wooden sticks per group and a piece of paper with lines on it.

Q1. In your groups, throw the wooden sticks randomly on the piece of paper and then fill in the summary table below.

Rough work (optional):

WOODEN STICK LANDS...	TALLY	TOTAL
Crossing a line		
Not crossing a line		
	Total no. of sticks:	

Summary Table:

TOTAL NO. OF STICKS (T)	TOTAL NO. OF STICKS THAT CROSS A LINE (C)	DISTANCE BETWEEN LINES IN MM (D)	LENGTH OF STICKS IN MM (L)

Using the results from his experiment with needles, Buffon estimated the value of using the following formula:

$$\pi \approx \frac{2L}{dx}$$

Where: **L** is the length of the needle (or sticks for us), **d** is the distance between the lines and **x** is the proportion of needles that cross the line (C/T).

Q2. Using this formula, what value did you get for π?

Q3. How accurate is your value for π?

Q4. How could you make your value for π more accurate? Any suggestions?

Q5. Could we ever measure an exact value for π? Explain your reasoning.

Numbers, Infinity and Shapes: Activity 3

Q1. Cut out the following two shapes (you may wish to sketch them first!)

(i) A shape with 2 faces and 4 edges

Sketch:

(ii) A shape with 2 faces and 1 edge

Sketch:

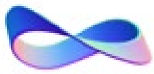
Q2. Challenge:

- Can you construct a shape with only 1 face and 1 edge using only paper, Sellotape and scissors?
- Can you prove that your shape only has 1 face and 1 edge using only a pencil?

Q3. Is this shape you constructed 2-D or 3-D?

- If it is 2-D it will have an area, can you measure it?
- If it is 3-D it will have a volume, can you measure it?
- What is the name of this shape? ________________

Numbers, Infinity and Shapes: Activity 4

The Möbius Strip 

Q1. Cut your strip in two, lengthwise along the centre. What do you predict will happen?

Q2. Make a new Möbius strip but this time, try cutting along one-third of its width. What do you predict will happen?

Q3. Make a strip with one full twist and cut it in two, lengthwise along the centre. What do you predict will happen?

The Story of Niamh

Niamh has been sent to live on the European Space Agency (ESA) space station, which has been constructed in the shape of a Möbius strip. The station has a central path on both the top and bottom surfaces of the loop. One afternoon, Niamh decides to find out just how long the ESA space station actually is and so, starting at a marker on the central path outside the control room, she turns and walks until she arrives back to where she began. Her health app says she has walked 2 kilometres.

Niamh therefore concludes that the space station must have 4 kilometres of central path in total since she knows that there is another path on the opposite surface of the loop.

What is Niamh's mistake? Please tick the relevant box.

- ☐ Niamh did not make any mistake - she is right!
- ☐ Niamh has miscalculated - there is actually 1 kilometre of walkway
- ☐ Niamh has miscalculated - there are actually 2 kilometres of walkway
- ☐ Niamh has miscalculated - there is actually an infinite length of walkway

Please explain your reasoning below:

Numbers, Infinity and Shapes: Activity 5

NUMBER OF HALF TWISTS	NUMBER OF FACES	NUMBER OF EDGES	WHEN CUT IN HALF
1			
2			
3			
4			
5			

Q1. Try making some strips with 2, 3, 4 or 5 half twists (i.e. 180° rotations). How many faces and edges does each one have? Cut these new strips in half and briefly describe what happens!

Q2. Do you notice any pattern?

Astronomy Mission

Introduction

Mathematics has many far-reaching applications, with some reaching as far as the stars. In fact, astronomy is one of the oldest sciences and has helped us to further our understanding of the universe. Astronomy is more than just creating a map of the observable stars, it is an important field of study which focuses on the building blocks of our universe. Fundamental mathematical techniques, such as trigonometry, enable us to uncover important astronomical phenomena and planetary information, including distances to celestial objects, orbital speeds, and even the mysterious force of gravity. This workshop will therefore provide students with the opportunity to use the mathematics they are already familiar with to study a new planet orbiting a distant star.

Aim of Workshop

The aim of this workshop is to show students the applicability of mathematical concepts, particularly in the realm of the natural sciences. Through working in groups to solve the problems arising from planning an interstellar mission, it is hoped that students will experience the collaborative nature that is central to STEM-based work. Moreover, this workshop aims to reintroduce mathematical topics such as scientific notation and trigonometry in an astronomical context, whilst also exploring new concepts such as force, parallax, and Kepler's laws of planetary motion. (This workshop may be particularly relevant for students interested in Physics and/or Applied Mathematics).

Learning Outcomes

By the end of this workshop students will be able to:

- Explain, in their own words, what is meant by parallax
- Convert large numbers such as distances/speeds to scientific notation
- Apply trigonometric ratios to find distances between planets and stars

Materials and Resources

Scale of the universe video clip (optional)

Keywords

Period
A period of motion is the length of time it takes a planet to complete a cycle of revolution about the Sun

Arc second
An arc second is 1/3600th of a degree. Just as there are 60 minutes in an hour and 60 seconds in a minute, a degree is divided into 60 arc minutes and an arc minute is divided into 60 arc seconds

Astronomical Unit (AU)
Unit of length used by astronomers for distances in space. One astronomical unit is equal to the average distance between the Earth and the Sun (approx. 150 million kilometres)

Astronomy Mission: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5 mins (00:05)	**Introduction to Astronomy**	– Mention the importance of mathematics in astronomy, particularly trigonometry (see **Workshop Introduction**). – Outline the aim of the workshop (to discover information about a new planet).
5 mins (00:10)	**Trigonometry Revision**	– You may wish to revise the trigonometric ratios sine, cosine, and tangent for right angle triangles.
5 mins (00:15)	**Parallax**	– Explain what is meant by parallax and ask students to try the Blink Test (see **Appendix – Note 1**). – Ask them what they noticed? Which finger appeared to move the most? – Class discussion on how parallax might help us determine how far away a star/planet is.
10 mins (00:25)	**Scale of the Universe and Scientific Notation**	– Brief video on the scale of the universe (see link in **Additional Resources**). – Mention the need for special notation to represent such large (and small) numbers. – You may wish to revise scientific notation with students (see **Appendix – Note 2**).
15 mins (00:40)	**Activity 1 The Expedition Begins**	– Hand out the formulae sheet to each pair of students. – **Activity Sheet 1**: In pairs, students calculate the distance to a new planet (see **Appendix – Note 3**).

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5 mins (00:45)	**Revision**	– Describe what is meant by force and mention Newton's second law of motion (see **Appendix – Note 4**). – Discuss the difference between weight and mass and ask students if they know the units for each (see **Appendix – Note 4**). – You may wish to recap the distance-speed-time relationships. – Emphasise the importance of distance and velocity having the same units e.g. **km** and **km/hr** or **m** and **m/sec** respectively.
10-15 mins (01:00)	**Activity 2 The New Planet**	– **Activity Sheet 2:** In pairs, students now attempt activity 2 using the formulae sheet (see **Appendix – Note 5**).
5 mins (01:05)	**Kepler's Laws**	– Discuss Kepler's laws of planetary motion and mention Newton's contributions (see **Appendix – Note 6**).
10 - 15 mins (01:20)	**Activity 3 The New Planet**	– **Activity Sheet 3**: Students now try to find the period of the new planet (see **Appendix – Note 7**). – Ask students if they noticed anything particular about the period of this new planet.

Astronomy Mission: Workshop Appendix

Note 1: The Blink Test and Parallax

Parallax is the apparent shift in the position of an object due to a change in the observer's point of view. The effect can be demonstrated by closing one eye and holding up your index finger at arm's length. Place your other index finger at a position that is closer to your face (see Figure 1). Now, look at your fingers with the other eye. As you alternate between your left and right eye, your fingers appear to move back and forth. This observation is due to the fact that the position from which you are viewing your finger is changing; a phenomenon known as parallax. Notice also that the finger closest to you appears to move the most relative to the background. When objects are too far away however, this shift in position is not apparent. This serves to demonstrate the limitations of approximating distances by parallax.

A further illustration of parallax (or triangulation) is shown in Figure 2 below. In this case, the tree is to the right of Person A and to the left of the Person B due to their different points of view. We can calculate how far away the tree is by using the angle of parallax and basic trigonometric calculations.

Figure 6: Blink test demonstration

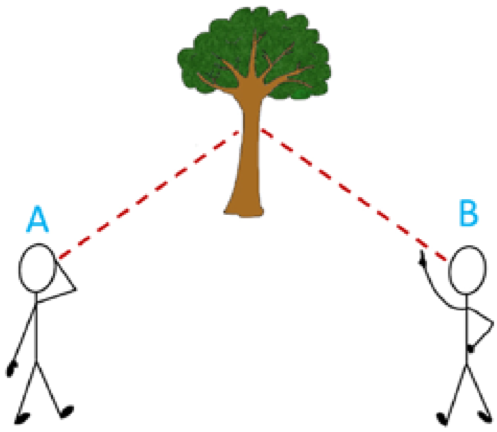


Figure 7: Example of Parallax

This same effect occurs when observing (presumed stationary) stars from earth as it orbits around the Sun. Some stars appear to move a lot, whilst others do not seem to move as much. Therefore, by measuring the parallax angle, it is possible to determine approximately how far away a celestial object is. Astronomers achieve this by viewing the star or planet from two different locations in the Earth's orbit around the Sun. For very distant stars, there does not appear to be any shift in position and consequently, it can be difficult to estimate their distance from the Earth.

Note 2: Scientific Notation

Formula for scientific notation: (number between 1 and 9) x 10^{power}

Example questions to ask students:

Q1. The closest distance between Earth and Pluto is 4,300,000,000 kilometres. How could we write this in scientific notation?

4.3×10^9 kilometres

Q2. What would 5326.6 be in correct scientific notation?

5.3266×10^3

Note 3: Solutions for Activity 1

Q1. Using your knowledge of trigonometry, find the distance from Proxima B to where the signal is coming from (blue planet).

We can bisect the parallax angle to form two right angled triangles.

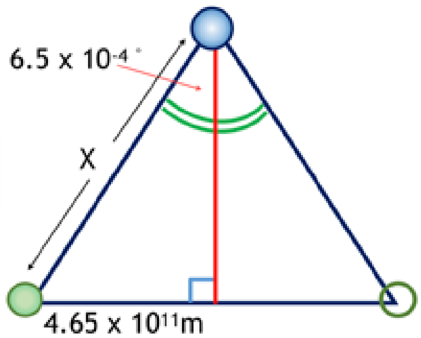
The angle at the top vertex is therefore:

$1.3 \times 10^{-3}/2 = 6.5 \times 10^{-4}\,°$

To find the desired distance x we can use: $\sin\theta = \dfrac{\textit{opposite}}{\textit{hypothenuse}}$

$$\sin(6.5 \times 10^{-4}) = \frac{(4.65 \times 10^{11})}{x}$$

$$x = \frac{(4.65 \times 10^{11})}{\sin(6.5 \times 10^{-4})} = 4.09885 \times 10^{16}$$



Q2. You are travelling at a speed of 1.5×10^8m/s (half the speed of light), how long will it take you to reach the new planet in seconds? (Hint: how are distance, speed and time related?)

Time = Distance/Speed
T = $4.09885 \times 10^{16} / 1.5 \times 10^8$ = 273256794.6 seconds

(Note: this answer may vary depending on the number of decimal places that are kept from the previous answer. The above answer was computed by using the ANS function on the calculator)

Q3. Can you calculate how many years it will take to reach the new planet?

60 secs in a min
60 mins in an hour
24 hours in a day
365 days in a year

60 x 60 x 24 x 365 = 315360000 seconds in a year
No. of years = 273256794.6/315360000
No. of years = 8.66 years

Note 4: Force, Mass and Weight

Force

Force is any interaction that, when unopposed, causes a change in the motion of an object. This idea is summarised by **Sir Isaac Newton** in his Laws of Motion. In fact, Newton's second law describes force as the mass of an object times its acceleration, more commonly denoted by the formula, F = ma. The SI unit for force is Newton (N), with 1 Newton equal to 1 kg x m/s^2.

Mass and Weight

The mass of an object is a measure of the amount of matter it contains, whereas its weight is the gravitational force that is exerted on it. Since an object or body will have the same composition regardless of its location, mass is independent of gravitational forces. An individual with a mass of 60kg on Earth, for example, would also have a mass of 60kg on the moon, given that they still contain the same amount of matter! Weight, on the other hand, is dependent on gravity and consequently, this same individual would weigh less on the moon since the force of gravity is far less on the moon than on Earth. In fact, weight is defined as the mass of an object times its acceleration due to gravity and since weight is a force, the SI unit is Newton (N).

Note 5: Solutions for Activity 2

Q1. What is the value of g on Planet Nua?

$W = mg$ $\qquad$ $g = 784/80$

$g = W/m$ $\qquad$ $g = 9.8 m/s^2$

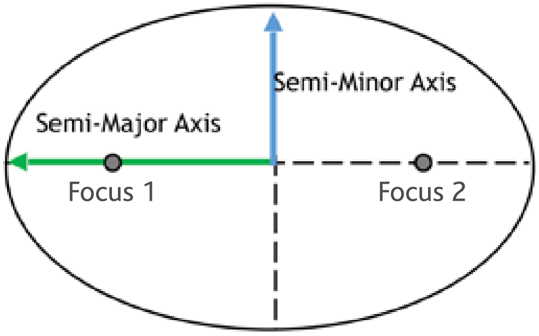
Q2. Calculate Newton's Gravitational constant (G).

$$G = \frac{gR^2}{M} = \frac{(7.15)\,(5.1\times10^7)^2}{2.79\times10^{26}} = 6.67 \times 10^{11}\ m^3\ kg^{-1}\ s^{-2}$$

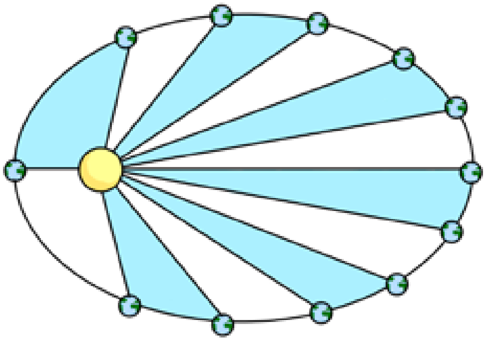
Note 6: Kepler's Laws of Planetary Motion

Johannes Kepler was a German mathematician and astronomer who devised three laws to describe the motion of planets orbiting the Sun, each of which is outlined below.

1. All planets move in elliptical orbits about the Sun. An ellipse is a regular oval shape, characterised by two foci, a minor axis, and a major axis. The sun is located at one of the two foci.



2. An imaginary line from the centre of the Sun to the centre of a planet will sweep out equal areas in equal intervals of time (see diagram below). The closer a planet is to the Sun, the stronger the Sun's gravitational pull and thus, the faster the planet's orbital speed.



3. Kepler's third law describes the relationship between a planet's distance from the Sun and the time it takes to complete its orbit (period). The larger the orbit, the longer it will take the planet to complete it given that the Sun's gravitational pull will be weaker when it is further away. This law is represented by the following formula:

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

Where **T** is the period, **a** is the semi-major axis, **M** is the mass of the planet, **G** is Newton's gravitational constant and π is our old constant friend pi.

Whilst Kepler described the motions of the planets, he did not provide any explanation of why the planets move in this way. In fact, Kepler's third law only applies to planets that orbit around the Sun but not other orbits such as the Moon's orbit around the Earth, for example. However, Sir Isaac Newton provided a more general explanation for the motions of the planets and other celestial objects, which is summarised in his Universal Law of Gravitation and Laws of Motion.

Note 7: Solutions for Activity 3

Q1. Assuming the phone signal is fibre optic (travels at the speed of light), calculate the distance between Planet Nua and Planet Eile (The value for the speed of light is on the formulae sheet).

3 minutes = 180 seconds. Using the distance formula, we get:

Distance = speed x time = $(3 \times 10^8 m/s)(180s) = 5.4 \times 10^{10}m$

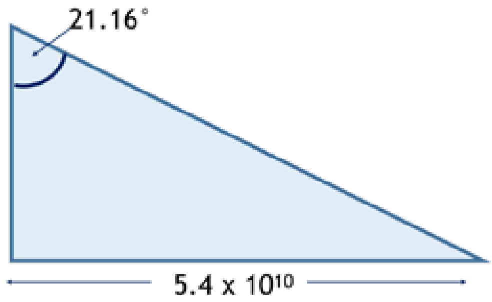
Q2. Another research team have found the angle between the vertices of Planet Nua, the star, and Planet Eile to be 21.16°. Find the distance from Planet Nua to the star (i.e. the semi-major axis).

We can use the formula:

$$\text{Sin}\theta = \frac{opposite}{hypothenuse}$$

$$x = \frac{5.4 \times 10^{10}}{\sin(21.16)}$$

$x = 1.49595 \times 10^{11}m$



Q3. Using the information you have calculated, find a value for the period of Planet Nua and change it from seconds to days. Do you notice anything significant about this figure? (Take M = 1.9891×10^{30} kilogrammes for the mass).

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

$a = 1.49595 \times 10^{11}m$
$G = 6.67 \times 10^{-11}\ m^3kg^{-1}s^{-2}$ (same for home planet)
$M = 1.9891 \times 10^{30}$

Substituting this into the formula above we get:

$T^2 = 9.96 \times 10^{14}$
$T = 31559467.68$
$T = 31559467.68/(60 \times 60 \times 24)$ days
$T = 365.3$ days...i.e. 1 year. The mystery planet is Earth!

Sources and Additional Resources

https://www.youtube.com/watch?v=uaGEjrADGPA (Scale of universe video)
http://www.physicsclassroom.com/class/circles/Lesson-4/Kepler-s-Three-Laws (Kepler's laws)

Astronomy Mission: Formulae Sheet

Distance	distance = speed x time
Newton's Law of Universal Gravitation	$F = \frac{GMm}{R^2}$
Gravitational Constant	$G = \frac{gR^2}{M}$
Force (Newton's Second Law)	$F = ma$
Weight	$W = mg$
Kepler's Third Law	$T^2 = \frac{4\pi^2 a^3}{GM}$

Note:

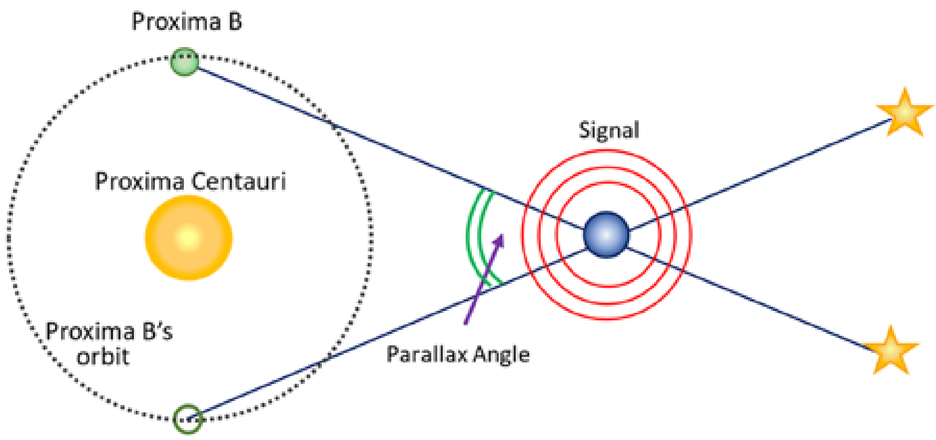
g = Acceleration due to gravity
M = Mass of planet
R = Radius of planet
G = Newton's gravitational constant
F = Force
T = Period
V = Velocity
a = Semi-major axis

Speed of Light C = 3×10^8 m/s

Astronomy Mission: Activity 1

The Expedition Begins

You are on planet Proxima B in an orbit around the star Proxima Centauri as shown below. The star Proxima Centauri is approximately 3.1 Astronomical Units (4.65×10^{11}m) from the centre of your planet. A signal has been detected from outer space, possibly from a planet which was, until now, undetected. Through methods of parallax, a research team have found that the Parallax Angle from the point of origin of the signal is 4.68 arc seconds
(or 1.3×10^{-3} degrees).



Q1. Using your knowledge of trigonometry, find the distance from Proxima B to where the signal is coming from (Blue planet). (Draw your blueprints)

Q2. You are travelling at a speed of 1.5×10^8m/s (half the speed of light), how long will it take you to reach the new planet in seconds? (Hint: how are distance, speed and time related?)

Q3. Can you calculate how many years it will take to reach the new planet?

Astronomy Mission: Activity 2

The New Planet

Congratulations you have discovered a new planet, which we will call Planet Nua!

You would like to find out how strong the force of gravity is on Planet Nua. Suppose your mass is 80kg. You step on a scale on the new planet and your weight reads 784N.

Q1. What is the value of g on Planet Nua?

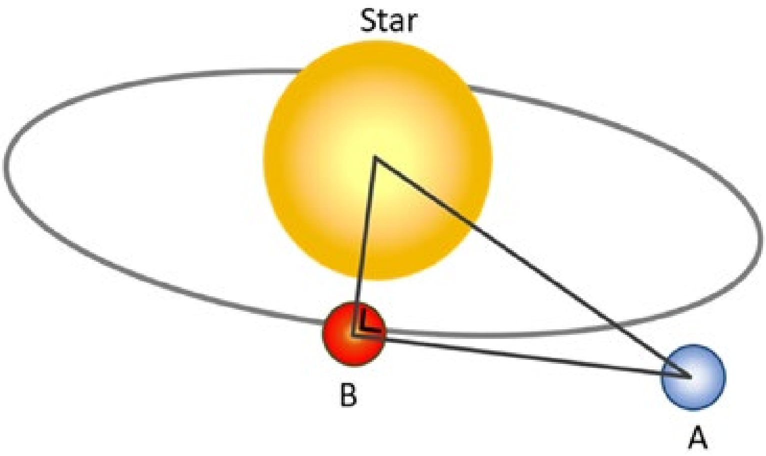
Q2. Taking values from your home planet to be

- $g = 7.15 m/s^2$
- $M = 2.79 \times 10^{26} kg$
- $R = 5.1 \times 10^7 m$

Calculate Newton's Gravitational constant (G). Note: G and g are different!

Astronomy Mission: Activity 3

You are on Planet Nua (A). You would like to know the distance from your planet to the nearest star. Your friends are chilling on Planet Eile (B) so you call them up. The time delay between you calling and your friends receiving the call is 3 minutes.



Q1. Assuming the phone signal is fibre optic (travels at the speed of light), calculate the distance between Planet Nua and Planet Eile (The value for the speed of light is on the formulae sheet).

Q2. **Another research team have found the angle between the vertices of Planet Nua, the star, and Planet Eile to be 21.16°. Find the distance from Planet Nua to the star (i.e. the semi-major axis).**

Q3. **Using the information you have calculated, find a value for the period of Planet Nua and change it from seconds to days. Do you notice anything significant about this figure?**

(Take $M = 1.9891 \times 10^{30}$ kilogrammes for the mass).

Logic

Introduction

Logic is an important concept which has a scope that spans a vast range of disciplines including philosophy, linguistics, computer science, and mathematics. Logic allows us to formulate arguments in a concise manner in order to make deductions and reach conclusions. As such, it is a very useful tool for mathematicians to have. In fact, **George Boole,** an English mathematician who was based in Co. Cork for much of his life, had a keen interest in logic and decided to develop a system for interpreting logical statements in a mathematical way, which eventually became known as Boolean algebra. This branch of algebra involves only two values for variables, true or false, which are subject to three basic operations: AND, OR, and NOT. While there are several additional Boolean operators, they can all be obtained from the composition of these simpler operations. While Boolean algebra was introduced by Boole in the 19th century, the practicality of Boolean algebra did not actually become apparent until a hundred years later when the Information Age began. In fact, it first found a practical application in representing the behaviour of switches in electronic circuits and is now used as the foundation of computer programming. It powers everyday digital devices such as our phones, tablets, laptops, and even search engines like Google! That is part of the beauty of mathematics - while it may sometimes seem an abstract form of ideas and research, the future holds the potential to find important uses for such mathematical discoveries.

Aim of Workshop

The aim of this workshop is to introduce students to the basic Boolean operators and their first practical application in circuit theory. The usefulness of logic as a tool for reasoning will also be emphasised, in addition to its importance in specific areas of mathematics including Boolean algebra.

Learning Outcomes

By the end of this workshop students will be able to:

- Explain the importance of logic in mathematics
- Describe, in their own words, how the AND, OR and NOT operators work
- Describe circuit configurations using Boolean logic statements
- Recognise the technological advances facilitated by Boolean algebra

Keywords

Logic
Logic represents the systematic study of the form of arguments

Logic gate
A logic gate is a circuit that regulates the flow of electrical current in a digital system by taking binary inputs and producing an output under Boolean operations

Logic: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5–10 mins (00:10)	**Activity 1 Who Robbed Kim?**	– Introduce the context of the task. – **Activity Sheet 1:** In pairs, students attempt to solve the mystery of who robbed Kim in Paris (see **Appendix – Note 1**). – Encourage students to draw a table to summarise the suspect's claims.
10 mins (00:20)	**Introduction to Logic Gates**	– Introduce the idea of logic and explain that it is a systematic study of the form of arguments. – Mention that logic has a wide range of applications (See **Workshop Introduction**). – Introduce Boolean algebra and logic gates. Discuss the AND, OR, and NOT gates and draw the symbol for each for students to see (see **Appendix – Note 2**). – You may wish to provide an example of the AND, OR, and NOT gate (see **Appendix – Note 3**).
10 mins (00:30)	**Activity 2 Truth Tables**	– **Activity Sheet 2**: Students fill in activity 2 using the AND, OR and NOT gates (see **Appendix – Note 4**).
10–15 mins (00:45)	**Activity 3 Circuit Tables**	– Describe what is meant by a circuit and explain how it works, making explicit reference to a switch being either open or closed. – **Activity Sheet 3**: Students complete activity 3 using the logic operators (see **Appendix – Note 5**).

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
15 mins (01:00)	**Activity 4 Passcode Riddle**	– **Activity Sheet 4:** In pairs, students try to solve the passcode riddle (see **Appendix – Note 6**). – For the extension question, encourage students to write out all the possible codes that start with 9 AND follow the rules. Now change the first digit to 8 and repeat. Ask students if they notice a pattern after completing several rows. – For Q2, encourage students to write down all possible codes with a product of 36. – If students are stuck, emphasise the fact that the friend didn't know the code after the second clue (suggesting that there are perhaps different codes which have the same sum).
5–10 mins (01:10)	**Boolean Algebra**	– You may wish to mention the use of Boolean algebra in search engines (see **Appendix – Note 7**). – You may also like to discuss Kay McNulty and her contributions to computer programming (see **Appendix – Note 8**).

Logic: Workshop Appendix

Note 1: Solutions to Activity 1

The following table summarises the suspects' claims regarding the robbing of Kim's ring.

	KANYE TELLS US	BEYONCÉ TELLS US	TRUMP TELLS US	MICHAEL D TELLS US	JAY-Z TELLS US
Kanye			✗		✗
Beyoncé	✓			✓	
Trump		✗		✓	
Michael D					✓
Jay-Z	✗	✗	✓		

✓ = robbed Kim ✗ = didn't robbed Kim

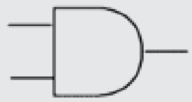
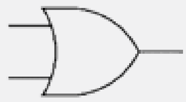
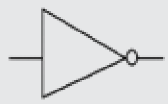
Beyoncé claims that "it wasn't Trump, it wasn't Jay-Z". Since one of these statements is a lie, we know that it must have been either Trump or Jay-Z who committed the crime. However, Michael D tells us that "it was Trump, it was Beyoncé", so we have narrowed it down to three suspects - Trump, Jay-Z or Beyoncé! Trump appears in both of these claims and looking at the rest of the suspects' reports, logic proves that Trump was in fact the one who robbed the ring!

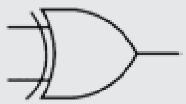
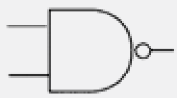
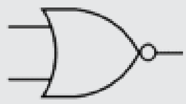
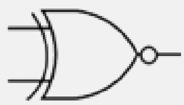
Note 2: Boolean Algebra and Logic Gates

George Boole was a self-taught English mathematician in the 1800s who was interested in extending the applicability of **Aristotle's** philosophical approach to logic. He therefore formulated a system for interpreting logical statements in a mathematical manner, which eventually became known as 'Boolean algebra' in his honour. This form of algebra only concerns the binary variables, true, and false, which are often simply denoted 1 and 0, respectively. These variables are subject to the basic Boolean operators AND, OR and NOT; each of which produces a single output.

Boolean logic is credited with laying the foundations for the information age given that it has been fundamental in the development of digital circuits. Logic gates, in particular, are important components of modern digital systems, which depend on the Boolean operators. These operations allow logic gates to control the flow of current within the system. Logic gates usually have two binary inputs - true and false - and produce a single output. A logic gate is therefore similar to a function in that it takes an input(s) and returns an output.

Different combinations of logic gates enable us to perform more complex operations within the circuit. The 7 basic logic gates are thus AND, OR, NOT, XOR, NAND, NOR, and XNOR, each of which is outlined on the following page. Whilst we will only cover the AND, OR and NOT gate in this workshop, you may still wish to discuss the other logic gates with your students.

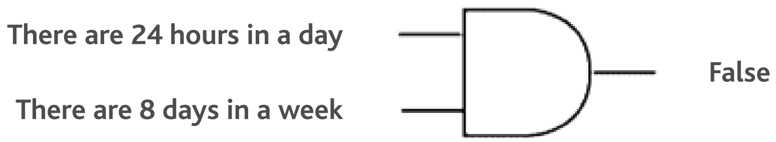
LOGIC GATE	DESCRIPTION	SYMBOL
AND	The AND gate is only true when the two inputs are true. Otherwise, the output is false.	
OR	The OR gate is true if at least one of the inputs is true. Otherwise, the output is false.	
NOT	The NOT gate simply negates the input. For example, if the input is true, the output will be false and vice versa. It is therefore commonly referred to as the inverter gate.	

LOGIC GATE	DESCRIPTION	SYMBOL
XOR	The XOR gate is similar to the OR gate. However, the output is true if either, but not both, of the inputs are true i.e. the output will be false if both inputs are false or if both inputs are true.	
NAND	The NAND gate functions as an AND gate followed by a NOT gate. So, if A was true and B was true, then A AND B would be true. However, since this is followed by a NOT gate, the final output is therefore false.	
NOR	The NOR gate operates as an OR gate followed by a NOT gate.	
XNOR	The XNOR gate is a XOR gate followed by a NOT gate.	

Note 3: Logic Gate Examples

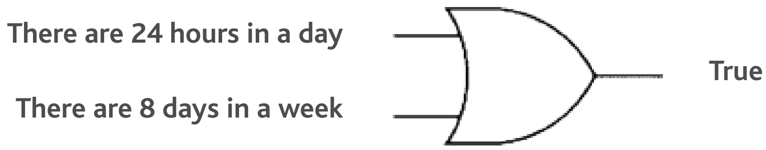
AND Gate

The following AND gate states "there are 24 hours in a day AND there are 8 days in a week". Since both of the claims in this statement are not true, the overall statement is considered false. In other words, since both inputs are not true, the output is therefore false.



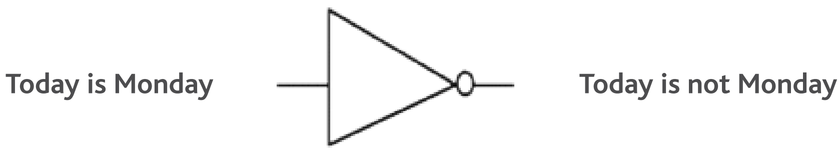
OR Gate

The following OR gate example states "there are 24 hours in a day OR there are 8 days in a week". Since one of these claims is true, the overall statement is considered true. In other words, since at least one of the inputs is true, the output is also true.



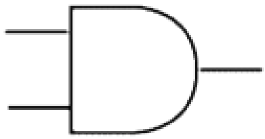
NOT Gate

The NOT gate simply negates the input. Therefore, if the input is "today is Monday", then the output would be "today is not Monday".



Note 4: Solutions for Activity 2

1. (i) What is the symbol for AND?



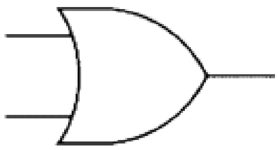
(ii) What is the output for each of the four scenarios below for an AND gate?

	A	B	A AND B
1	TRUE	FALSE	FALSE
2	FALSE	TRUE	FALSE
3	TRUE	TRUE	TRUE
4	FALSE	FALSE	FALSE

(iii) In your own words, how would you define the AND operation?

AND outputs true only when both inputs are true

2. (i) What is the symbol for OR?



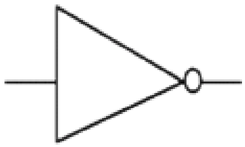
(ii) What is the output for each of the four scenarios below for an OR gate?

	A	B	A OR B
1	TRUE	FALSE	TRUE
2	FALSE	TRUE	TRUE
3	TRUE	TRUE	TRUE
4	FALSE	FALSE	FALSE

(iii) In your own words, how would you define the OR operation?

OR outputs true when either of the two inputs is true

3. (i) What is the symbol for NOT?



(ii) What is the output for each of the two scenarios below for a NOT gate?

	A	NOT A
1	TRUE	FALSE
2	FALSE	TRUE

(iii) In your own words, how would you define the NOT operation?

NOT changes the value of a true input to false and false input to true (i.e. it inverts the input value).

Note 5: Solutions for Activity 3

Q1. (i) Switch A and switch B are said to be in series. Based on your knowledge of a closed circuit, when do you think the bulb will light up?

When A and B are closed

(ii) Summarise this situation using one of the logic operators we mentioned earlier.

A AND B

Q2. Now switch A and switch B are said to be in parallel. Using a logic operator, describe when the bulb will light up in Circuit 2.

A OR B

Q3. (i) Which switch must always be closed for the bulb to light?

Switch R

(ii) In addition to having all of the switches closed, what other combinations of closed switches will turn on the bulb?

a) R and P

b) R and Q

(iii) This situation can be represented by logic gates. Fill in the blanks below with P, Q and R to see how this works.

R AND (P OR Q)

Q4. (i) In addition to having all of the switches closed, what other combinations of closed switches will turn on the bulb?

a) D

b) N_1 and N_2

(ii) Represent this scenario with logic gates (Look back at question 3 (iii) if you're stuck!)

(N_1 AND N_2) OR D

(iii) Fill in the blanks with your answer from the previous question, then complete the table.

N_1	N_2	D	N_1 AND N_2	(N_1 AND N_2) OR D	BULB LIGHT?
T	T	T	T	T	YES
T	T	F	T	T	YES
T	F	T	F	T	YES
F	T	T	F	T	YES
F	F	T	F	T	YES
T	F	F	F	F	NO
F	F	F	F	F	NO

Note 6: Solutions for Activity 4

Q1. How many possible 3-digit codes could be made with 3 digits?

Since there are 10 possible digits (0, 1, …, 9) for each of the three code numbers we get:

10 x 10 x 10 = 1000

Extension: try this question with all four rules

Method 1:

One way to approach this extension task is to set the first digit of your code equal to 9. Now write down all the possible codes that start with 9 AND obey the four rules. Once you have written down all possible codes, change the first digit to 8 and repeat (see below).

999

888 889 899

777 778 779 788 789 799

666 667 668 669 677 678 679 688 689 699

As we change the first digit of the code for each row, we notice the following pattern emerge:

FIRST DIGIT OF CODE	NO. OF CODES STARTING WITH THE RESPECTIVE DIGIT THAT ALSO OBEY THE FOUR RULES	DIFFERENCE BETWEEN ROWS
9	1	+2
8	3	+3
7	6	+4
6	10	+5
5	15	+6
4	21	+7
3	28	+8
2	36	+9
1	45	+10
0	55	

The values 1, 3, 6, 10, 15, 21, 28, 36, 45, 55 follow the pattern of the triangular sequence. Adding these 10 values together, we get 220 possible codes that obey all four rules.

Method 2:

Alternatively, we could fix the first digit and let the middle digit equal 9. We know there is only one possibility for the third digit in this case, namely 9. Hence, regardless of what the first digit is, there is only one possibility for the third digit when the middle digit is 9.

Continuing this process for a fixed first digit, with 8 as the second digit, there are just two possibilities for the third digit, namely 8 or 9. Thus, we can tabulate all possible 3-digit codes which obey all four rules.

N=	9	8	7	6	5	4	3	2	1	0	SUM
9	1	1	1	1	1	1	1	1	1	1	= 10 X 1 = 10
8		2	2	2	2	2	2	2	2	2	= 9 X 2 = 18
7			3	3	3	3	3	3	3	3	= 8 X 3 = 24
6				4	4	4	4	4	4	4	=7 X 4 = 28
5					5	5	5	5	5	5	= 6 X 5 = 30
4						6	6	6	6	6	= 5 X 6 = 30
3							7	7	7	7	= 4 X 7 = 28
2								8	8	8	= 3 X 8 = 24
1									9	9	= 2 X 9 = 18
0										10	= 1 X 10 = 10
											TOTAL: 220

Also, notice 220 = $\binom{n+2}{3}$ for n = 10, the 10th tetrahedral number.

For more on the tetrahedral numbers see https://en.wikipedia.org/wiki/Tetrahedral_number

Q2. What is the required code?

Of all the positive numbers that have 36 as their product, only two have the same sum (it has to be one of these because otherwise the friend could have cracked it after the second clue!) These are 9 x 2 x 2 and 6 x 6 x 1. Since the largest number only appears once the answer is: 9 x 2 x 2. However, it needs to be written from smallest to largest by code rules: 2 2 9

Note 7: Boolean Algebra and Search Engines

Search engines such as Google or Yahoo employ the use of Boolean logic in order to refine search queries, thereby returning the most relevant pages. The AND operator, for example, will only return results which contain all of the specified terms, whereas the OR operator will return results which contain either one, several or all of the search terms. For instance, if you search "apples or oranges", you will get pages with "apples", pages with "oranges", and also pages with both terms. The NOT operator, on the other hand, will remove any pages with the specified word, allowing you to filter out unwanted results. However, search engines do not actually recognise "not" as the operation, but rather the minus sign character (-). If you are searching for squares but do not want any red squares, for example, then you would have to search "squares -red".

Note 8: Kay McNulty

Whilst Boolean algebra played a significant role in the development of computer programming and digital systems, several Irish mathematicians also made profound contributions to the area. **Kay McNulty**, for example, was an Irish-born mathematician who left Donegal at the age of three to live in Pennsylvania, USA. She graduated with a mathematics degree from Chestnut Hill College in 1942 before securing a job with the U.S. Army research laboratory. This position involved calculating the trajectories for bullets and shells, which was crucial information for soldiers using artillery guns during the war. Later, she was one of six women transferred to work on the E.N.I.A.C. (Electronic Numerical Integrator and Computer) computer; the world's first general purpose digital computer. This mammoth machine was the size of a railway carriage and filled most of the room in which she worked. It had approximately 3,000 switches, 18,000 vacuum tubes, and industrial-sized fans to cool the various components. This huge computer could calculate the trajectories in a matter of seconds, but it was the women's responsibility to use a range of switches and plugs in order to determine the sequence required for these calculations. Unfortunately, they did not have manuals or instructions on how to operate the computer and therefore had to figure out how it worked themselves. Through their work, the women advanced the whole field of computing, with some of the techniques they developed still being used today!

Sources and Additional Resources

http://whatis.techtarget.com/definition/logic-gate-AND-OR-XOR-NOT-NAND-NOR-and-XNOR

https://academo.org/demos/logic-gate-simulator/ (Simulator)

Logic: Activity 1

Logic Times

Bringing you the top activities since 2015 | 3rd April

Who Robbed Kim in Paris

Several suspects have been brought forward regarding the robbing of Kim Kardashian in Paris last week. Each suspect told one truth and one lie, but we don't know which is which. Their claims are reported below:

Image credit: huffingtonpost.com

Kanye: 'It wasn't Jay-Z, it was Beyoncé'

Beyoncé: 'It wasn't Trump, it wasn't Jay-Z'

Trump: 'It was Jay-Z, it wasn't Kanye'

Michael D: 'It was Trump, it was Beyoncé'

Jay-Z: 'It was Michael D, it wasn't Kanye'

Can you figure out who robbed Kim based on the suspects' claims?

Logic: Activity 2

Truth Tables

Q1. AND

(i) What is the symbol for AND?

(ii) What is the output for each of the four scenarios below for an AND gate?

	A	B	A AND B
1	TRUE	FALSE	
2	FALSE	TRUE	
3	TRUE	TRUE	
4	FALSE	FALSE	

(iii) In your own words, how would you define the **AND** operation?

Q2. OR

(i) What is the symbol for OR?

(ii) What is the output for each of the four scenarios below for an OR gate?

	A	B	A OR B
1	TRUE	FALSE	
2	FALSE	TRUE	
3	TRUE	TRUE	
4	FALSE	FALSE	

(iii) In your own words, how would you define the **OR** operation?

Q3. NOT

(i) What is the symbol for NOT?

(ii) What is the output for each of the two scenarios below for a NOT gate?

	A	NOT A
1	TRUE	
2	FALSE	

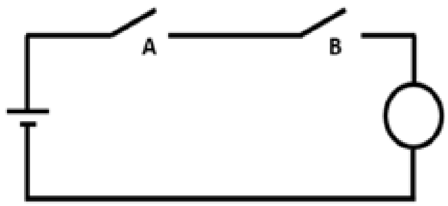
(iii) In your own words, how would you define the **NOT** operation?

The AND, OR and NOT truth tables below will come in handy as you progress through the workshop!

Logic: Activity 3

Using the terms from the previous activity, complete the following questions on circuits.

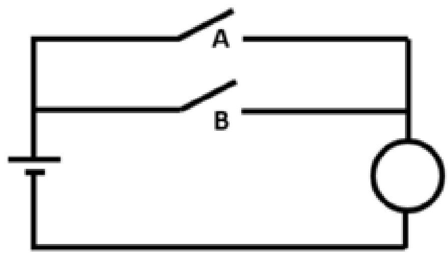
Circuit 1



Q1. (i) Switch A and switch B are said to be *in series*. Based on your knowledge of a closed circuit, when do you think the bulb will light up?

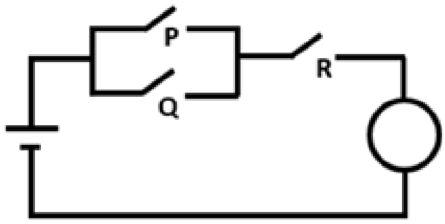
(ii) Summarise this situation using one of the logic gate operators we mentioned earlier.

Circuit 2



Q2. Now switch A and switch B are said to be *in parallel*. Using a logic operator, describe when the bulb will light up in Circuit 2.

Circuit 3



Q3. (i) Which switch must **always** be closed for the bulb to light?

(ii) In addition to having all of the switches closed, what other combinations of closed switches will turn on the bulb?

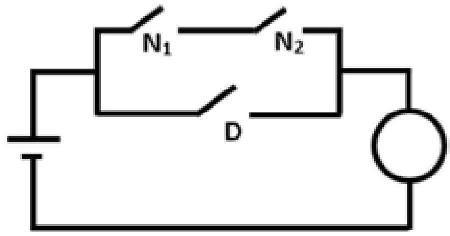
a:________________

b:________________

(iii) This situation can be represented by logic gates. Fill in the blanks with P, Q and R to see how this works.

____ AND (___ OR ___)

Circuit 4



Q4. (i) In addition to having all of the switches closed, what other combinations of closed switches will turn on the bulb?

a:________________

b:________________

(ii) Represent this scenario using logic gates (Look back at Q3 (iii) if you're stuck!)

(iii) The table below lists the possible configurations for the circuit. True corresponds to a closed switch, whilst False means the switch is open. Fill in the blanks with your answer from the previous question, then complete the table.

N_1	N_2	D	N_1 _______ N_2	(N_1 _______ N_2) _____ D	BULB LIGHT?
T	T	T			
T	T	F			
T	F	T			
F	T	T			
F	F	T			
T	F	F			
F	F	F			

(iv) Does your answer in the final column agree with your reasoning in Q4 (i)? Why?

Logic: Activity 4

A safe in a bank is secured with a passcode and protected by a guard. You and your friend need access to the safe. The bored guard decides to help you out with a few logic clues, under the agreement that you each get one guess and that only one of you is led to the safe, whilst the other listens over a walkie talkie. You must both be correct, or the guard will have you removed from the premises.

He firstly outlines the rules of the passcode:

1. A digit is a whole number from 0 to 9
2. The passcode contains 3 digits
3. The second digit is greater than or equal to the first
4. The third digit is greater than or equal to the second

Q1. Taking only the first two rules of the passcode, how many possible codes could be made with 3 digits? (Extension: try this question with all four rules)

You remain at the entrance to the bank with the walkie talkie so that you can hear the conversation of your friend and the guard. Your friend is led through the bank and down one of the corridors to where the safe is kept. She then receives the following clues:

- The product (the three digits multiplied) of the code's digits is 36
- The sum of the code's digits is the same as the corridor number where the safe is held (your friend knows the corridor number, but you do not!)
- The largest of the digits appears only once in the code

Your friend could not crack the code after the second clue. However, when given the third clue, she immediately cracks it and enters in the code. The guard also demands the passcode from you through the walkie talkie.

Q2. What is the required code?

(Hint: Write out all the possible 3-digit codes with a product of 36)

UCD Undergraduate Student Volunteers for Maths Sparks 2017

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Participating Schools

Cabinteely Community College, Cabinteely

Caritas College, Ballyfermot

Greenhills College, Walkinstown

Killinarden Community School, Tallaght

Kylemore College, Ballyfermot

St. Aidan's Community School, Tallaght

St. Dominic's Secondary School, Ballyfermot

St. Mark's Community School, Tallaght

St. Paul's Secondary School, Greenhills

St. Tiernan's Community School, Balally

Alexandra College, Milltown

Acknowledgements

We would like to thank the post–primary students who attended the series of workshops in 2017. Their curiosity, energy and feedback have made the programme an incredibly enjoyable and rewarding experience and we wish them well in all of their future endeavours. We would also like to thank the schools, teachers, principals, career guidance counsellors, school communities, parents and guardians for supporting students in taking part in this extra-curricular mathematics programme.

We would like to sincerely thank all of the UCD students who volunteered their time to design, trial, and conduct these workshops. Their interest and passion for mathematics has been a key feature of the programme and has been a positive influence on the learning experiences of participating post-primary students.

We would sincerely like to thank Emily Lewanowski-Breen for her work in compiling, editing and illustrating this booklet.

We are tremendously grateful to the staff of the UCD Access and Lifelong Learning Centre who liaised with schools and post-primary students participating in the workshops. We would like to express our gratitude to Áine Murphy, without whom the project would not have succeeded.

We would like to thank all of the staff of the UCD School of Mathematics and Statistics who worked with our student volunteers, participated in the organisation and presentation of workshops, and continue to encourage post–primary students to enjoy and see value in the subject of Mathematics. We would like to thank our Head of School, Prof. Gary McGuire, and Dean of the College of Science, Prof. Joe Carthy, for their continued support of this initiative and for their encouragement in organising programmes to promote science and mathematics with communities inside and outside of UCD.

Finally, we would like to thank SFI Discover for their funding support in expanding and developing this programme and the UCD SPARC initiative for the initial funding in beginning this project.

Dr Aoibhinn Ní Shúilleabháin
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