UCD Maths Enrichment: The ‘double counting trick’ for combinatorial problems.

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1 Bipartite graphs

A graph is a set of vertices connected by edges. Figure 1 gives two examples of graphs.

For some counting-type problems, it can help to organise data using what’s called a *bipartite* graph. A bipartite graph is just a graph where the vertices are split into two different groups and the edges only connect vertices from one group to the other, i.e., there is never an edge between two vertices of the same group. The graph on the right-hand side in figure 1 has a bipartite grouping.

It is often very convenient to represent a finite bipartite using a rectangular grid of numbers, called its matrix, with rows corresponding to one set of vertices and columns corresponding to the other set of vertices. For example the following matrix represents the bipartite graph on the right-hand side of

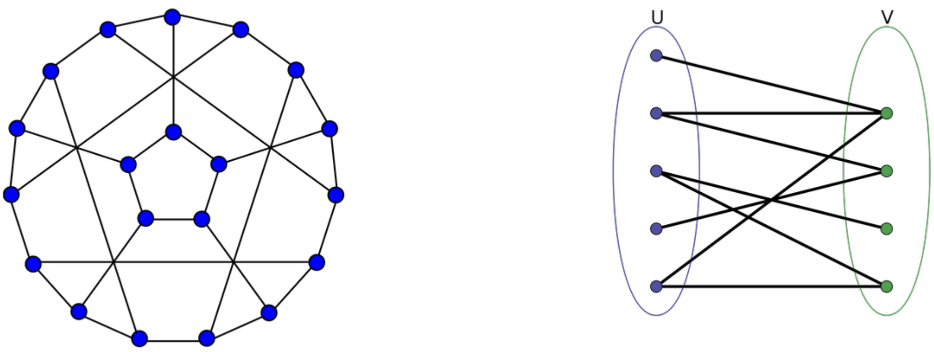

Figure 1: the graph on the right has a bipartite grouping

figure 1, taking the five vertices grouped as U for rows and the four vertices grouped as V for columns:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Sometimes the edges naturally come with labels (often called colours) and this fits nicely with the matrix representation.

Useful questions to ask yourself when faced with counting problems like the ones discussed below are:

1. What am I being asked to count?
2. If I were to represent the things I am being asked to count as the edges of a bipartite graph, what would be the rows and columns of its matrix?

If you manage to provide answers to the above then the 'trick' is that you can often make progress by noting that a row-wise count is the same as a column-wise count.

2 Handshaking lemma

Suppose we have a graph G with E edges and vertices $v_1, v_2, \ldots, v_n$, then:

$$d(v_1) + d(v_2) + \cdots + d(v_n) = 2E, \tag{1}$$

where $d(v_i)$ is the number of edges through vertex v_i.

rows: vertices of G

columns: edges of G

3 Counting solutions

Example 3.1. *Suppose that 17 contestants took part in a mathematics competition with 9 problems. Each problem was solved by exactly 11 contestants. Show that there exists a pair of contestants who, between them, solved all 9 problems*[1].

[1]These examples are based on `https://hcmop.wordpress.com/2012/03/30/`

rows: problems
columns: unordered pairs of contestants

Exercise 3.2. *Suppose that c contestants took part in a mathematics competition with p problems, and that each problem was solved by at least k contestants. Show that there exists a pair of contestants who, between them, solved all p problems provided:*

$$p < \frac{c(c-1)}{d(d-1)},$$

where $d = c - k$.

Example 3.3. *Suppose that 15 contestants took part in a mathematics competition with 15 problems. Each contestant solved exactly 6 problems. Show that there exists a pair of contestants who solved at least 3 problems in common.*

Exercise 3.4. *Suppose that c contestants took part in a mathematics competition with p problems. Each contestant solved at least s problems, and p divides evenly into cs. Show that there exists a pair of contestants who solved at least k problems in common, for some:*

$$k \geq \frac{s(cs-p)}{p(c-1)}.$$

Exercise 3.5. *What about when p does not divide evenly into cs?*

4 IMO 1998 Q2

Example 4.1. *In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either 'pass' or 'fail'. Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $k/a \geq (b-1)/(2b)$.*

rows: contestants
columns: unordered pairs of judges

Exercise 4.2. *State and prove the corresponding statement if b is even.*

It is useful to look at the results of this question to see how contestants performed when the exam took place back in 1998. The vast majority of contestants scored either 0 or 7! Surely this is because each contestant either knew the 'double counting trick' or they didn't[2].

[2]Knowing how contestants performed can be quite useful when practicing with past IMO questions. Try exploring http://olivernash.org/2017/08/05/visualising-imo-results/

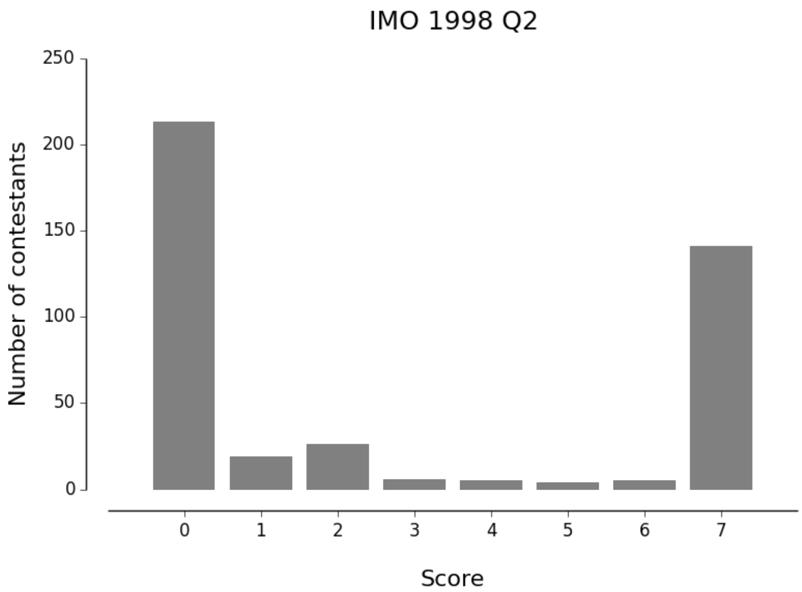

Figure 2: how many contestants got each score between 0 and 7

5 IMO 2001 Q3

Example 5.1. *Twenty-one girls and twenty-one boys took part in a mathematical contest.*

- *Each contestant solved at most six problems.*
- *For each girl and each boy, at least one problem was solved by both of them.*

Prove that there was a problem that was solved by at least three girls and at least three boys.

rows: girls
columns: boys

Exercise 5.2. *Suppose that a girls and a boys took part in mathematical contest and that:*

- *each each contestant solved at most n problems,*
- $n > 1$,
- *for each girl and each boy, at least one problem was solved by both of them.*

Prove that there was a problem solved by at least x girls and at least x boys for some:

$$x \geq \frac{a}{2(n-1)}.$$

Exercise 5.3. *Suppose that g girls and b boys took part in mathematical contest and that:*

- *each girl solved at most n problems,*
- *each boy solved at most m problems,*
- *for each girl and each boy, at least one problem was solved by both of them.*

Prove that there was a problem solved by at least x girls and at least x boys for some:

$$x \geq \frac{gb}{gn + bm - \min(gn, bm, g+b)}.$$

Exercise 5.4. *State and prove the corresponding statement if we wish to find a problem solved by at least x girls and at least y boys. Sketch the results in the x-y-plane.*

6 IMO 2005 Q6

Example 6.1. *In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.*

rows: contestants
columns: unordered pairs of problems

7 Average number of divisors

For a natural number n, let $d(n)$ be the number of divisors of n, and let

$$\bar{d}(n) = \frac{1}{n}\left(d(1) + d(2) + \cdots d(n)\right),$$

be the average of the number of divisors of all natural numbers up to n. Then[3] we have:

$$\bar{d}(n) \sim H_n \sim \ln n.$$

where:

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n},$$

is the n^{th} harmonic number (in fact we can be more precise).

rows: the numbers $1, 2, \ldots, n$

columns: the numbers $1, 2, \ldots, n$

8 Higher dimensions

Exercise 8.1. *Suppose we have a d-dimensional cube of natural numbers of shape:*

$$100 \times 400 \times 900 \times \cdots \times (10d)^2,$$

and any line parallel to an axis of the cube contains at most 21 different natural numbers. Show that the cube contains a number which appears at least 4 times in each of the d lines passing through it.

You may use without proof the fact that $\pi^2 < 10$.

This is really a generalisation of IMO 2001 Q3. For $d = 2$, it is close to the original problem but don't be confused: it's just a coincidence the number 21 appears in both statements and it plays a different role.

[3] This example is taken from: Aigner M., Ziegler G.M., 'Proofs from THE BOOK', third edition, chapter 22, section 4, pages 142–143.