MATHEMATICAL ENRICHMENT / OLYMPIAD TRAINING

www.irmo.ie 14/10/2023 PROFESSOR GARY MCGUIRE
U.C.D.

INEQUALITIES

BASIC INEQUALITY : $x^2 \geq 0$ equality holds if and only if $x=0$

(x is a real number ~~with $x \geq 0$~~)

Let's apply this to $(x-y)^2 \geq 0$ where $x \geq 0$, $y \geq 0$ are any real numbers

$$x^2 - 2xy + y^2 \geq 0$$

$$x^2 + y^2 \geq 2xy$$

equality holds if and only if $x-y=0$
$x=y$

Example Show that

$$x^2+y^2+z^2 \geq xy + yz + xz$$

x, y, z are real numbers

Solution:

$$x^2+y^2 \geq 2xy$$

$$y^2+z^2 \geq 2yz$$

$$x^2+z^2 \geq 2xz$$

Add $2x^2+2y^2+2z^2 \geq 2xy+2yz+2xz$

Divide by 2.

Example Show $\boxed{\dfrac{a^2}{a+b} \geq \dfrac{3a-b}{4}}$ where $a>0, b>0$.

we know $a^2+b^2 \geq 2ab.$

$$a^2 \geq 2ab - b^2$$

Add $3a^2$ $\quad 4a^2 \geq 3a^2 + 2ab - b^2$

$$= (3a-b)(a+b)$$

Divide by 4 and $a+b$ $\quad \dfrac{a^2}{a+b} \geq \dfrac{3a-b}{4}$

Rough work

$$4a^2 \geq (3a-b)(a+b)$$

$$= 3a^2 + 2ab - b^2$$

$$a^2 \geq 2ab - b^2$$

$$a^2 + b^2 \geq 2ab$$

(2)

Example Show $\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a} \geq \frac{a+b+c}{2}$ $\begin{pmatrix} a>0 \\ b>0 \\ c>0 \end{pmatrix}$

Solution

$$\frac{a^2}{a+b} \geq \frac{3a-b}{4}$$

$$\frac{b^2}{b+c} \geq \frac{3b-c}{4}$$

$$\frac{c^2}{c+a} \geq \frac{3c-a}{4}$$

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a} \geq \frac{3a-b}{4}+\frac{3b-c}{4}+\frac{3c-a}{4}$$

$$= \frac{2a+2b+2c}{4}$$

$$= \frac{a+b+c}{2}.$$

Ir. M.O. 1999

Q8 Given $a>0, b>0, c>0, d>0$, and $a+b+c+d=1$,

Show $\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a} \geq \frac{1}{2}$.

Solution. we know

$$\frac{a^2}{a+b} \geq \frac{3a-b}{4}$$

$$\frac{b^2}{b+c} \geq \frac{3b-c}{4}$$

$$\frac{c^2}{c+d} \geq \cancel{\frac{3d-c}{4}}\ \frac{3c-d}{4}$$

$$\frac{d^2}{d+a} \geq \frac{3d-a}{4}$$

Add $\frac{a^2}{a+b}+\cdots+\frac{d^2}{d+a} \geq \frac{a+b+c+d}{2} = \frac{1}{2}$

The arithmetic mean of x and y is $\frac{x+y}{2}$

The geometric mean of x and y is $\sqrt{xy}$

The AM–GM inequality says

$$\boxed{\frac{x+y}{2} \geq \sqrt{xy}}$$

$x > 0$
$y > 0$.

$(x + y \geq 2\sqrt{xy})$

equality holds if and only if $x = y$.

$\underline{\text{proof}}$: $(\sqrt{x} - \sqrt{y})^2 \geq 0$

$$x - 2\sqrt{x}\sqrt{y} + y \geq 0.$$

$$x + y \geq 2\sqrt{xy}$$

$$AM \geq GM.$$

<u>Example</u> If $x > 0$ show $x + \frac{1}{x} \geq 2$ and equality holds iff $x = 1$

<u>Solution</u> In AM - GM inequality put $y = \frac{1}{x}$

$$\frac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} = \sqrt{1} = 1$$

Multiply by 2.

eq. holds iff $x = \frac{1}{x}$
$x^2 = 1 \Rightarrow x = 1$

<u>Example</u> If $x, y > 0$ show $\frac{x}{y} + \frac{y}{x} \geq 2$

<u>Solution</u> Apply AM–GM inequality to $\frac{x}{y}$ and $\frac{y}{x}$

get $$\frac{\frac{x}{y} + \frac{y}{x}}{2} \geq \sqrt{\frac{x}{y} \cdot \frac{y}{x}} = 1$$

Example Show $(a+b)(b+c)(c+a) \geq 8abc$ $\begin{pmatrix} a>0 \\ b>0 \\ c>0 \end{pmatrix}$

Solution $a+b \geq 2\sqrt{ab}$ by AM-GM

$b+c \geq 2\sqrt{bc}$ "

$c+a \geq 2\sqrt{ca}$ "

Multiply $(a+b)(b+c)(c+a) \geq 8\sqrt{ab \cdot bc \cdot ca} = 8abc$

The AM-GM inequality for n numbers is

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

equality holds iff $a_1 = a_2 = \cdots = a_n$

Example Show $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$ $\quad x>0$, $y>0$, $z>0$

Apply AM-GM with $n=3$ to $\frac{x}{y}, \frac{y}{z}, \frac{z}{x}$

Get $$\frac{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}{3} \geq \sqrt[3]{\frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}} = 1$$

Example Show $x^6 + y^6 + 4 \geq 6xy$ $\quad x>0$, $y>0$.

Solution: apply AM-GM $(n=6)$ to $x^6, y^6, 1, 1, 1, 1$

$$x^6 + y^6 + 1 + 1 + 1 + 1 \geq 6\sqrt[6]{(x^6)(y^6)(1)(1)(1)(1)}$$
$$= 6xy$$

Example Show that $x^2 + \frac{4}{x} \geq 3\sqrt[3]{4}$ $\quad x>0$

Solution: Apply AM-GM to $x^2, \frac{2}{x}, \frac{2}{x}$

(5) Get $x^2 + \frac{2}{x} + \frac{2}{x} \geq 3 \cdot \sqrt[3]{x^2 \cdot \frac{2}{x} \cdot \frac{2}{x}}$

$= 3\sqrt[3]{4}$

notice $x^2 + \frac{1}{x} + \frac{3}{x} \geq 3 \sqrt[3]{x^2 \cdot \frac{1}{x} \cdot \frac{3}{x}} = 3 \cdot \sqrt[3]{3}$ not as good

equality holds iff $x^2 = \frac{2}{x}$

$x^3 = 2$

$x = \sqrt[3]{2}$

Example Given $a+b+c=1$, show $a>0$, $b>0$, $c>0$.

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.$$

equality holds iff $a=b=c=\frac{1}{3}$.

The Harmonic Mean of $a_1, a_2, \ldots, a_n$ is

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

The AM-HM inequality says: AM $\geq$ HM

$(a_i > 0)$

proof

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

Multiply this. Get n 1's from $a_1 \cdot \frac{1}{a_1}$

Also get $\frac{a_1}{a_2} + \frac{a_2}{a_1}$ This is ≥ 2.

Get $\frac{n(n-1)}{2}$ terms $\frac{a_i}{a_j} + \frac{a_j}{a_i}$ $i \neq j$

So product is $\geq n + 2 \cdot \frac{n(n-1)}{2}$

$= n^2$

(6) So $(a_1+\cdots+a_n)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right) \geq n^2$ use either

or $\frac{a_1+\cdots+a_n}{n} \geq \frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}$.

Example $(n=3)$ if $a+b+c=1$,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3^2 = 9.$$

Example Show $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \geq \frac{3}{2}$ $a>0$, $b>0$, $c>0$

want to use AM-HM with $a+b$, $b+c$, $a+c$

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} = \frac{a+b+c}{b+c}+\frac{a+b+c}{a+c}+\frac{a+b+c}{a+b}-3$$

$$= (a+b+c)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right)-3$$

$$= \frac{1}{2}\cdot(2a+2b+2c)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right)-3$$

use AM-HM $\geq \frac{1}{2}\cdot 9 - 3$

$$= \frac{3}{2}$$

Cauchy-Schwarz Inequality.

$$(a^2+b^2)(x^2+y^2) = (ax+by)^2 + (ay-bx)^2$$

$$\geq (ax+by)^2$$

Cauchy Schwarz for $n=2$.

equality holds iff $ay-bx=0$

$$\frac{a}{x}=\frac{b}{y}$$

$n=3$ (7) $(a^2+b^2+c^2)(x^2+y^2+z^2) \geq (ax+by+cz)^2 \leftarrow$ C.S. ineq.

Example Given $x>0,\ y>0,\ z>0, \quad xyz=1,$

show $x+y+z \leq x^2+y^2+z^2.$

In Cauchy-Schwarz put $a=1,\ b=1,\ c=1$

get $3(x^2+y^2+z^2) \geq (x+y+z)^2$

Apply AM-GM to x,y,z: $\quad x+y+z \geq 3\sqrt[3]{xyz} = 3$

$\Longrightarrow \quad 3(x^2+y^2+z^2) \leq (x+y+z)(x^2+y^2+z^2)$

$\Longrightarrow \quad (x+y+z)^2 \leq (x+y+z)(x^2+y^2+z^2)$

Divide by $x+y+z$

For n numbers, Cauchy-Schwarz is

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

$$(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2) \geq (a_1b_1+\cdots+a_nb_n)^2$$

See also Chebychev's Inequality.

Titu's Lemma.

Schur's inequality.

Example $a>0,\ b>0,\ c>0.$ Show

$$a+b+c \leq \frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}$$

Solution

$$(a+b+c)^2 = \left(\frac{a}{\sqrt{c}}\cdot\sqrt{c} + \frac{b}{\sqrt{a}}\cdot\sqrt{a} + \frac{c}{\sqrt{b}}\cdot\sqrt{b}\right)^2 \underset{\text{by C-S}}{\leq} \left(\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}\right) \times (c+a+b)$$

Divide by $a+b+c$.