# Mathematical Enrichment — Oct. 26th, 2024

## Kevin Hutchinson — Number Theory

ie. "whole numbers" ... $-2, -1, 0, \underbrace{1, 2, 3, 4, \ldots}$

integers

natural numbers.

"

positive integers.

**Example**

I.MO 2003 Problem 6

Show that for each prime $p$, there exists a prime $q$ such that $n^p - p$ is not divisible by $q$ for any positive integer $n$.

---

**Example** Find all integer solutions of

$$x^2 + y^2 = z^2$$

---

Water pouring problems


Pour 4 litres exactly

Solution 1: $3 \cdot 3 - 1 \cdot 5 = 4$ ✓

Solution 2: $2 \cdot (5 - 3) = 4$ ✓

---

Obtain 1 litre: (a) Given 4, $4 - 3 = 1$

or (b) $2 \cdot 3 - 5 = 1$

Pour 57 litres exactly:

$$57 \cdot (2 \cdot 3 - 5) = 57$$

$$(57 \cdot 2) \cdot 3 - 57 \cdot 5 = 57.$$

---

(b) $\boxed{5}$ $\boxed{17}$

Pour exactly 1 litre.

$$3 \cdot 17 - 10 \cdot 5 = 1$$

or $$7 \cdot 5 - 2 \cdot 17 = 1$$

---

(c) $\boxed{4}$ $\boxed{18}$

Pour 3 litres?

Impossible! $4m \pm 18n$ will always be even.

---

(d) $\boxed{35}$ $\boxed{49}$

Pour <u>3</u> litres?

Since $7 \mid 35$ and $7 \mid 49$ any number of the form $35m \pm 49n$ (*) will also be a multiple of <u>7</u>.

We say 7 is a <u>common divisor</u> of 35, 49.

---

(*) If $d \mid a$ and $d \mid b$ then $d \mid a \pm b$.

Rephrase general problem as a pure mathematics problem:

Given (positive) integers $m, n$.

When can we measure out (or obtain) the quantity $l$ ?

i.e. Do there exist integers $x, y$ such that

$$\boxed{x \cdot \underline{m} \pm y \cdot \underline{n} = \underline{l}}$$

i.e. Solve for integers $x, y$

(If we can solve, we'll say $l$ is "obtainable")

Note 1 If $l$ is obtainable, then so is $tl$ for any integer $t$:

$$\underbrace{tx}\cdot m + \underbrace{ty}\cdot n = t \cdot l.$$

Note 2 If $d$ is a common divisor of $m, n$ and if $l$ is obtainable, then $d$ is a divisor of $l$ also ($d \mid l$)

In particular, if $g = \gcd(m, n)$ then $g \mid l$ if $l$ is obtainable.

Question But: Is $g$ obtainable?

Answer Yes, by Euclid's algorithm

(Euclid ca 350 BC.)

## Basic principle

Given $m, n$ if $m = tn + r$

for any integers $t, r$

then $\gcd(m,n) = \gcd(n,r)$

[Why? If $d \mid m$ and $d \mid n$ then $d \mid r = m - tn$.

If $d \mid n$ and $d \mid r$ then $d \mid m = tn + r$ ]

**Example** $35, 49.$

$$\left[\begin{array}{l} 49 = 1\cdot 35 + 14 \\ 35 = 2\cdot 14 + 7 \end{array}\right. \qquad \gcd(35,14) = \gcd(14,7).$$

$$(14 = 2\cdot 7 + 0)$$

$$7 = 35 - 2\cdot 14$$

$$= 35 - 2\cdot(49-35) = 3\cdot 35 - 2\cdot 49$$

**Example** $703, 1007$

Find $g = \gcd(703, 1007)$. Solve $703x + 1007y = g$

**Solution**

$$1007 = 703 + 304$$

$$703 = 2\cdot 304 + 95$$

$$304 = 3\cdot 95 + 19$$

$$(95 = 5\cdot 19 + 0)$$

$$19 = 304 - 3\cdot 95$$

$$= 304 - 3\cdot(703 - 2\cdot 304) = 7\cdot 304 - 3\cdot 703$$

$$= 7\cdot(1007 - 703) - 3\cdot 703$$

$$\boxed{19 = 7\cdot 1007 - 10\cdot 703}$$

<u>Theorem</u> Given nonzero integers $m$, $n$

If $g = \gcd(m,n)$ then we can solve

$$mx + ny = g$$

in integers $x, y$.

<u>Corollary</u> If $d$ is <u>any</u> common divisor of $m$, $n$ then $d \mid g$ where $g = \gcd(m,n)$.

---

<u>Exercises</u> (1) $\boxed{437}$ $\boxed{986}$ what amounts are obtainable?

Show how to pour $g = \gcd(437, 986)$.

(2) Find $\gcd(2^8+1, 2^{32}+1) = g$ and express $g$ as $x\cdot(2^8+1) + y\cdot(2^{32}+1)$ for some integers $x, y$.

(3) <u>IMO Romania 1959</u>

Prove that the fraction $\dfrac{21n+4}{14n+3}$ is <u>irreducible</u> for every natural number $n$.

kevin.hutchinson@ucd.ie