Mathematical Enrichment 18th Feb

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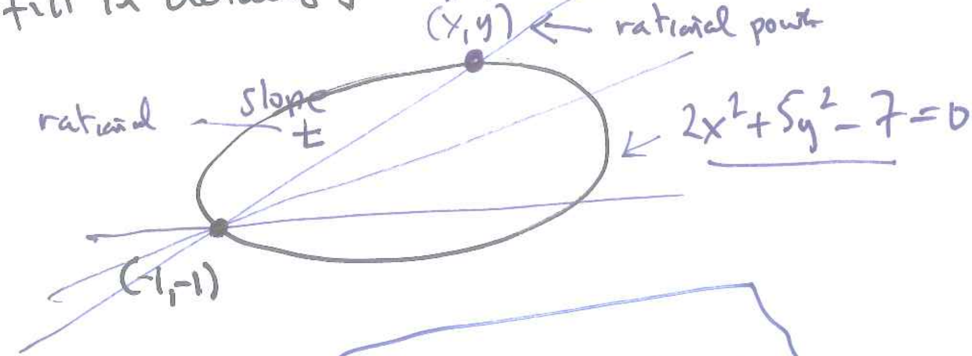
1. Find all rational solutions to

$$2x^2 + 5y^2 = 7$$

Find all integer solutions to

$$2a^2 + 5b^2 = 7c^2$$

Solution [Sketch - fill in details] $x(t), y(t)$ L_t



L_t: $y+1 = t\cdot(x+1)$: $\boxed{y = tx + (t-1)}$ quadratic in x; $x=-1$ is a root.

$$2x^2 + 5[tx + (t-1)]^2 - 7 = 0$$

$$x^2(5t^2+2) + x\cdot 10t(t-1) + \underbrace{5(t-1)^2 - 7}_{= 5t^2 - 10t - 2}$$

other root is

$$\left(x = -\frac{5t^2-10t-2}{5t^2+2},\quad y = \frac{5t^2+4t-2}{5t^2+2} \right)$$ t any rational number

eg choose $t=2$ $(x,y) = \left(\frac{2}{22}, \frac{26}{22}\right) = \left(\frac{1}{11}, \frac{13}{11}\right)$

So $2\cdot\left(\frac{1}{11}\right)^2 + 5\cdot\left(\frac{13}{11}\right)^2 = 7$

$$\Rightarrow \quad 2\cdot 1^2 + 5\cdot 13^2 = 7\cdot 11^2$$

Write $t = \frac{m}{n}$ $\quad n \neq 0$, m, n integers.

Then

$$(x, y) = \left(\frac{2n^2 + 10nm - 5m^2}{5m^2 + 2n^2}, \frac{5m^2 + 4mn - 2n^2}{5m^2 + 2n^2} \right)$$

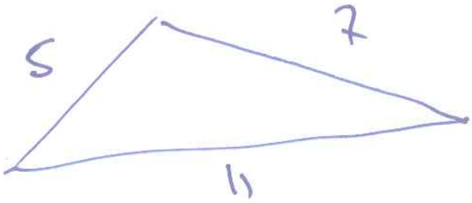
The general integer solution to $2a^2 + 5b^2 = 7c^2$

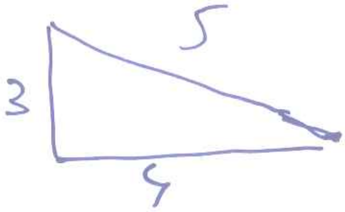
is

$$(a, b, c) = (2n^2 + 10nm - 5m^2;\ 5m^2 + 4mn - 2n^2;\ 5m^2 + 2n^2)$$

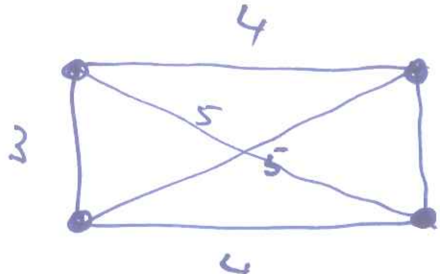
Find N points in the plane, not all collinear, such that all distances are integers.

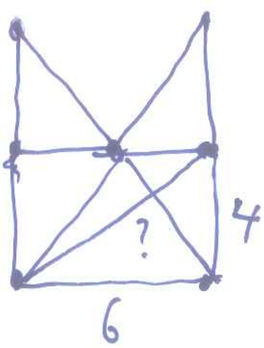
$\underline{N=3}$



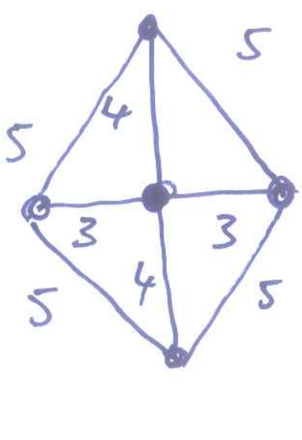


$N=4.$





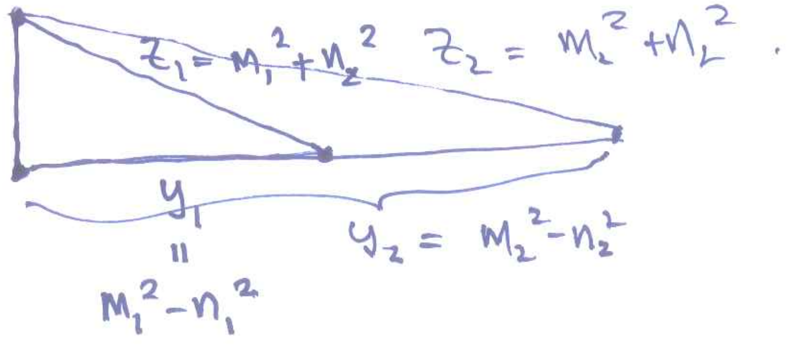
$\underline{N=5}$



A Solution

$x^2 + y^2 = z^2$ if $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$

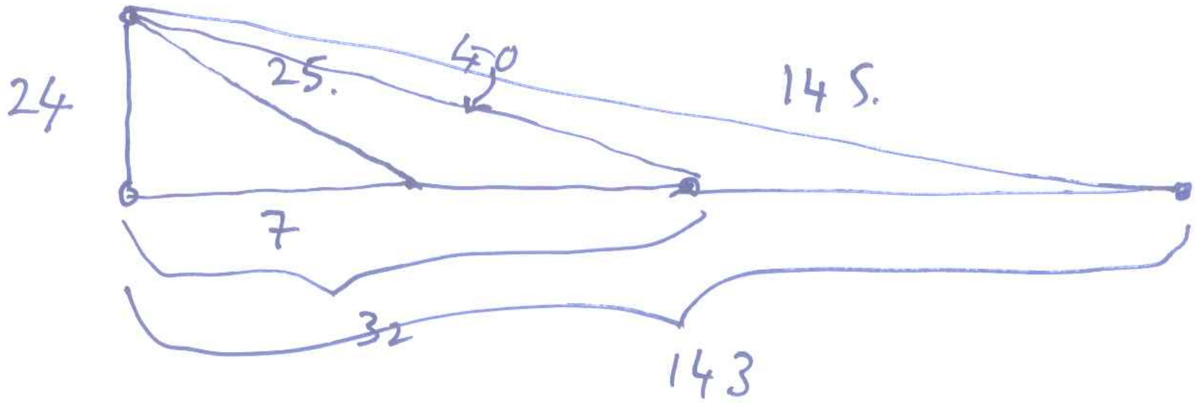
$$2m_1n_1 = 2m_2n_2 = x$$



Example

$$x = 2 \cdot 12 = \begin{array}{l} 2 \cdot (1 \cdot 12) \\ 2 \cdot (2 \cdot 6) \\ 2 (3 \cdot 4) \end{array}$$

$y_1 = 4^2 - 3^2 = 7$
$y_2 = 6^2 - 2^2 = 32$
$y_3 = 12^2 - 1^2 = 143$



$x = 2 \cdot l$ where l has at least N different factorizations.

Primes and Factorizations

The primes numbers $2, 3, 5, 7, 11, 13, \ldots$

Fundamental Theorem of Arithmetic

Every integer greater than 1 factors is one and only one way as a product of powers of distinct prime numbers:

$$7 = 7^1$$

$$15 = 3 \cdot 5 = 3^1 \cdot 5^1$$

$$16 = 2^4$$

$$48 = 2^4 \cdot 3$$

$$60 = 2^2 \cdot 3 \cdot 5$$

$$36 = 2^2 \cdot 3^2$$

In general if $n > 1$

$$\boxed{n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_t^{a_t}}$$

where $p_1 < p_2 < \ldots p_t$ primes

and $a_1 \geq 1, a_2 \geq 1, \ldots, a_t \geq 1$

Suppose d is a divisor of n.

So $n = d\,e$ for some integer e.

$$d = p_1^{b_1} \cdots p_t^{b_t} \qquad e = p_1^{c_1} \cdots p_t^{c_t} \qquad (b_i, c_i \geq 0)$$

$$\Rightarrow \quad (p_1^{b_1} \cdots p_t^{b_t}) \cdot (p_1^{c_1} \cdots p_t^{c_t}) = p_1^{a_1} \cdots p_t^{a_t}$$

$$p_1^{b_1+c} \cdot p_2^{b_2+c_2} \cdot \cdots \cdot p_t^{b_t+c_t} = p_1^{a_1} \cdots p_t^{a_t}$$

By <u>uniqueness</u> part of Fund. Thm,

$$b_1+c_1=a_1, \;\ldots,\; b_t+c_t = a_t.$$

$$(d = p_1^{b_1} \cdots p_t^{b_t})$$

In particular $0 \le b_1 \le a_1 \leftarrow a_1+1$ choices for b_1

$0 \le b_2 \le a_2 \leftarrow a_2+1$ choices for b_2

$\vdots$

$0 \le b_t \le a_t \leftarrow a_t+1$ choices for b_t.

$\therefore$

$\tau(n) \doteq$ # of divisors of n is $(1+a_1)\cdot(1+a_2)\cdots(1+a_t)$

<u>Example</u> What is $\tau(10!)$?

$$10! = 2\times3\times4\times5\times6\times7\times8\times9\times10$$

$$= 2^8\cdot3^4\cdot5^2\cdot7^1 = p_1^{a_1}\, p_2^{a_2}\, p_3^{a_3}\cdot p_4^{a_4}$$

$$\therefore\ \tau(10!) = (1+8)\cdot(1+4)\cdot(1+2)\cdot(1+1) = 270.$$

Do the same for $20!$

Divisors of $36 = 2^2 \cdot 3^2$

$\tau(36) = 3 \cdot 3 = 9$

$$\begin{array}{ccc} 1 & 3 & 3^2 \\ 2 & 2\cdot 3 & 2\cdot 3^2 \\ 2^2 & 2^2\cdot 3 & 2^2\cdot 3^2 \end{array}$$

Sum of all ~~the~~ divisors:

$$1+3+3^2 + 2\cdot(1+3+3^2) + 2^2\cdot(1+3+3^2)$$

$$= (1+2+2^2)\cdot(1+3+3^2)$$

$n = p_1^{a_1} p_2^{a_2}$ $\qquad \tau(n) = (a_1+1)\cdot(a_2+1)$

List the divisors

$$\begin{array}{cccc} 1 & p_2 & \cdots & p_2^{a_2} \\ p_1\cdot 1 & p_1\cdot p_2 & \cdots & p_1\cdot p_2^{a_2} \\ & \vdots & & \\ p_1^{a_1}\cdot 1 & p_1^{a_1} p_2 & \cdots & p_1^{a_1}\cdot p_2^{a_2} \end{array}$$

Then

Sum of all divisors of n is

$\sigma(n) =$

$$(1+p_2+\cdots+p_2^{a_2}) + p_1\cdot(1+p_2+\cdots+p_2^{a_2}) + \cdots +$$

row 1 row 2

$$\cdots + p_1^{a_1}\cdot(1+p_2+\cdots+p_2^{a_2})$$

row a_1+1

$$= (1+p_1+\cdots+p_1^{a_1})\cdot(1+p_2+\cdots+p_2^{a_2})$$

(Recall: If $p \neq 1$, $a > 1$ then

$$(1+p+p^2+\cdots+p^a)\cdot(p-1) = p^{a+1}-1$$

(check).

$$\therefore \quad 1+p+\cdots+p^a = \frac{p^{a+1}-1}{p-1}.\ )$$

$$\sigma(n) = \frac{p_1^{a_1+1}-1}{p_1-1}\cdot\frac{p_2^{a_2+1}-1}{p_2-1}.$$

In general, if $n = p_1^{a_1}\cdots p_t^{a_t}$ (Proof by induction on t)

$$\sigma(n) = (1+p_1+\cdots+p_1^{a_1})\cdot(1+p_2+\cdots+p_2^{a_2})\cdots(1+p_t+\cdots+p_t^{a_t})$$

$\parallel$ Sum of divisors

$$= \frac{p_1^{a_1+1}-1}{p_1-1}\cdot\frac{p_2^{a_2+1}-1}{p_2-1}\cdots\frac{p_t^{a_t+1}-1}{p_t-1}$$

$$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$$

$$\therefore \sigma(10!) = \frac{2^9 - 1}{2-1} \cdot \frac{3^5 - 1}{3-1} \cdot \frac{5^3 - 1}{5-1} \cdot (7+1)$$

$$= (2^9 - 1) \cdot \left(\frac{3^5 - 1}{2}\right) \cdot \frac{124}{4} \cdot 8 \quad \text{etc.}$$

$\tau(n)$ and $\sigma(n)$ are examples of arithmetic functions

$$\tau(ab) \neq \tau(a) \cdot \tau(b) \quad \text{in general.}$$

$$\tau(3) = 2$$

$$\tau(3^2) = 3 \neq \tau(3) \cdot \tau(3).$$

$$\tau(p_1^{a_1} \cdots p_t^{a_t}) = \underbrace{\tau(p_1^{a_1})}_{1+a_1} \cdot \underbrace{\tau(p_2^{a_2})}_{a_2+1} \cdots \tau(p_t^{a_t})$$

More generally, $\tau(ab) = \tau(a) \cdot \tau(b)$

provided a and b have greatest common divisor 1.

A function $f(n)$ of n (positive integers) is an arithmetic function if has the property that

$$f(nm) = f(n) \cdot f(m) \quad \text{whenever } n, m \text{ have gcd } 1.$$

<u>Theorem</u> If f is an <u>arithmetic</u> <u>function</u> then

$$f(p_1^{a_1} \cdots p_t^{a_t}) = f(p_1^{a_1}) \cdot f(p_2^{a_2}) \cdots f(p_t^{a_t}).$$

<u>Some problems to think about</u>

1. p a prime. $N >> p$. Show that the precise power of p which divides $N!$ is

$$\left[\frac{N}{p}\right] + \left[\frac{N}{p^2}\right] + \left[\frac{N}{p^3}\right] + \cdots$$

($[x]$ denotes the <u>integer part</u> of x.

$[5.7] = 5$
$[\pi] = 3$
$[e] = 2.$)

2. Calculate $\sigma(509)$, $\sigma(1128)$, $\sigma(20!)$

3 Prove that $\tau(n)$ is odd if and only if n is a <u>perfect square</u>.

4. a) Prove that if $3 \mid n+1$ then $3 \mid \sigma(n)$

b) [Putnam competition].

Prove that $24 \mid n+1$ then $24 \mid \sigma(n)$.

The number n is said to <u>perfect</u> if it is the sum of its (proper) divisors

i.e. if $\sigma(n) = 2n$

<u>Examples</u>

$$6 = 1 + 2 + 3$$

$$28 = \cancel{1 + 14 + 2 + 28}$$
$$1 + 7 + 2 + 14 + 4$$

<u>Euclid</u> (Suppose p is prime and) suppose $q = 2^p - 1$ is prime then

Mersenne prime

$$N = 2^{p-1} \cdot (2^p - 1) = 2^{p-1} \cdot q$$

is a perfect number [Exercise.]

<u>Examples</u>

$p = 2$, $q = 3. = 2^2 - 1$ $N = 2 \cdot 3 = 6$

$p = 3$, $q = 2^3 - 1 = 7$ $N = 4 \cdot 7 = 28$

$p = 5$, $q = 2^5 - 1 = 31$, $N = 2^4 \cdot 31 = 16 \cdot 31$

$\vdots$

<u>1700s</u>

<u>Euler proved</u> (difficult!) that if N is an <u>even</u> perfect number then

$$N = 2^{p-1}(2^p - 1)$$

Mersenne prime.

<u>Open problem</u>:
Are there any <u>odd</u> perfect numbers?