IMO 2021 Q1

Andrew D Smith
University College Dublin

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1 Problem: Avoiding Sums of Squares

1.1 Simplified Version of IMO 2021 Q1

Ivan writes the numbers 100, 101, 102, ... 200 each on different cards. He then shuffles these 101 cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

1.2 Brute Force Solution

Start dealing cards to avoid putting pairs in the same pile that add to a square, and see where we get stuck. The sum of two distinct cards lies between 201 and 399. Squares in that range are: 225, 256, 289, 324, 361.

Call the two piles A and B.

Step 1: Deal 100 in pile A.

Step 2: Deal 125, 156, 189 in pile B.

Step 3: Deal 131, 133, 135, 164, 168, 172, 199 in pile A.

Step 4: Deal 117, 121, 123, 152, 154, 158, 160, 162, 191, 193, 197 in pile B.

Step 5. Deal 102, 104, 108, 127, 129, 135, 137, 139, 164, 166, 168, 170, 172 in pile A.

Now we are stuck because 127 and 129 are both in pile A, but $127 + 129 = 256 = 16^2$.

We found this by trial and error. The problem setter likely did this. It was hard work. Is there a neater way?

1.3 Remarks

We have shown that the dealing of cards starting with card 100 gets stuck if we avoid pairs adding to squares. That is enough to answer the problem.

But could this method of proof fail? Could we hypothetically have tried dealing from 100, and not get stuck, but the claimed result could still hold?

Yes, it could fail, because maybe our process never uses up all the original 101 cards.

2 Search for a Simpler Solution

2.1 Try a Smaller Example

Might we find a proof in a smaller example we can generalise? Sometimes IMO questions refer to a large number (eg the year of the contest) but the result is true for any sufficiently large number. Suppose instead Ivan has cards numbered 10 to 20. Is it still impossible to avoid square sums?

Possible squares for the sum of two cards are are 25 or 36.

Step 1: Deal 10 and 15 in different piles.
Step 2: Deal 11 and 14 in different piles.
Step 3: Deal 12 and 13 in different piles.
Step 4: Deal 16 and 20 in different piles.
Step 5: Deal 17 and 19 in different piles.
Step 6: Deal 18 in either pile.

There are many ways to avoid square sums. How many different ways?

2.2 Reducing the Counterexample

Return to the original problem with cards numbered 100 to 200 inclusive. We had a counterexample in 5 steps. How can we simplify the example to its bare bones?

Working back through the calculations, we did not need to start with 100. We can start the story putting 164 in pile A (was step 3). That implies 160 and 197 are in pile B, which in turn puts

127 and 129 back in pile A.

So we have a cycle 164, 160, 129, 127, 197, 164 ... where every two consecutive terms add to a square.

As the cycle has an ODD period of 5 steps, we cannot avoid dealing two consecutive terms into the same pile.

Our calculations also reveal another odd cycle: 164, 160, 129, 127, 162, 199, 125, 164 ... with period 7.

2.3 Is there a Simpler Counterexample?

We found counterexample cycles of size 5 and 7. It suffices to find any odd cycle whose consecutive terms add to squares.

So maybe there is a 3-cycle to find? That would prove something we now know to be true, but would be simpler.

At this point, we reach for the back of an envelope. We are not proving anything yet; we are coming from the other end with a hunch for a proof and seeing if we can make it work. Our search might not work, that is, the claim could still be true even if there are no 3-cycles. So this is a long shot.

Suppose our 3-cycle is $x, y, z, x, \ldots$. Without loss of generality

$$100 \leq x < y < z \leq 200$$

There must be integers a, b, c such that:

$$
\begin{aligned}
x + y &= a^2 \\
x + z &= b^2 \\
y + z &= c^2
\end{aligned}
$$

with $a < b < c$. This is equivalent to:

$$
\begin{aligned}
x &= \frac{a^2 + b^2 - c^2}{2} \\
y &= \frac{a^2 + c^2 - b^2}{2} \\
z &= \frac{b^2 + c^2 - a^2}{2}
\end{aligned}
$$

Can we find such a, b, c? Let us refine the search.

The squares a, b, c are either all even numbers, or two odd numbers and one even number. Otherwise we fail to get integers x, y, z.

Furthermore, as x and z are between 100 and 200, we must have:

$$200 \leq b^2 - (c^2 - a^2) < b^2 + (c^2 - a^2) \leq 400$$

This means b must be between 15 and 19.

Furthermore,

$$0 < c^2 - a^2 < \min\{b^2 - 200, 400 - b^2\}$$

This puts a bound on the gap between a and c.

Also, as $a < b < c$ are integers we have $a \leq b-1$ and $c \geq b+1$ and therefore $c^2 - a^2 \geq 4b$

Check the cases:

b	$c^2 - a^2$		
	min	max	Parity
15	60	25	N/A
16	64	56	N/A
17	68	89	Not OK
18	72	76	OK
19	76	39	N/A

So the only solution is $a = 17, b = 18, c = 19$. This gives the 3-cycle 126, 163, 198, 126 ...

To answer the original question, we can merely pull the 3-cycle 126, 163, 198, 126 ... out of a hat. We can throw away the envelope on which we found the cycle.

So far, this gives us just another way of solving the original problem. Yet another reason why Ivan cannot avoid having two cards in the same pile adding to a square. But the 3-cycle is easier to generalise.

3 IMO Problem

3.1 IMO 2021 Q1

Let $n \geq 100$ be an integer. Ivan writes the numbers n, $n+1$, ... , $2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

3.2 Proof for $99 \leq n \leq 126$

The 3-cycle 126, 163, 198, 126 ...already proves impossibility.

3.3 Generalising the Counterexample

We look for other 3-cycles where $x < y < z$ lie within a range $z \leq 2x$.

To keep x, y, z in a narrow range, we must have a, b, c in a narrow range. The narrowest they can be is consecutive. As integers x, y, z imply we cannot just one odd number out of a, b, c, they must be odd, even, odd so that:

$$
\begin{aligned}
a &= 2k - 1 \\
b &= 2k \\
c &= 2k + 1
\end{aligned}
$$

so that:

$$
\begin{aligned}
x &= \frac{a^2 + b^2 - c^2}{2} = 2k(k - 2) \\
y &= \frac{a^2 + c^2 - b^2}{2} = 2k^2 + 1 \\
z &= \frac{b^2 + c^2 - a^2}{2} = 2k(k + 2)
\end{aligned}
$$

This triple provides a proof of impossibility provided that:

$$n \leq 2k(k-2) \leq 2k(k+2) \leq 2n$$

or, equivalently:

$$k(k+2) \leq n \leq 2k(k-2)$$

Setting $k = 9$ solved the original problem (to prove there must be two cards in the same pile adding to a square) for $99 \leq n \leq 126$. For other values of k we then have proofs for the following ranges of n:

k	Proven values of n		Overlap with next?
	min	max	
6	48	48	FALSE
7	63	70	FALSE
8	80	96	FALSE
9	99	126	TRUE
10	120	160	TRUE
11	143	198	TRUE

If we can prove that the overlaps continue for larger k, then we have solved the IMO problem for all $n \geq 100$.

The condition for case k to overlap with the next case is that:

$$2k(k-2) \geq (k+1)(k+3)$$

This is equivalent to:

$$k^2 - 8k - 3 = (k-9)(k+1) + 6 \geq 0$$

This of course holds for $k \geq 9$. The original claim then holds for all $n \geq 99$.

This is a valid solution to the IMO problem. It is rather lengthy, but would still earn full marks. We will now tidy it up a little to find the model solution.

Remark: We have answered the question posed. But what of the values of n where we have no proof? For example $n = 98$ lies in the gaps between intervals for $k = 8$ and $k = 9$. We have not proved that the result holds for $n = 98$, but it might still be true, just needing a more cunning proof.

It turns out the result fails when $n = 98$. See if you can find a way to split cards numbered from 98 to 196 (inclusive) into two piles so that no two in the same pile add to a square.

4 How to Write the Proof in 2 Pages

- Step 1: Given $n \geq 99$, let $(k+1)^2$ be the largest perfect square that does not exceed $n+1$, so that:

$$(k+1)^2 \leq n+1 < (k+2)^2$$

  As $n \geq 99$ then $k \geq 9$.

- Step 2: Show that:

$$n \leq 2k(k-2) < 2k^2+1 < 2k(k+2) \leq 2n$$

  so that each of $\{2k(k-2), 2k^2+1, 2k(k+2)\}$ appears on one of the cards numbered from n to $2n$. See below for proof.

- Step 3: Note that:

$$\begin{aligned} 2k(k-2) + 2k^2 + 1 &= (2k-1)^2 \\ 2k(k-2) + 2k(k+2) &= (2k)^2 \\ 2k^2 + 1 + 2k(k+2) &= (2k+1)^2 \end{aligned}$$

  At least two of $\{2k(k-2), 2k^2+1, 2k(k+2)\}$ must appear in the same pile, and add to a square number. This is what we had to prove.

It remains only to prove the inequalities in step 2.

We start with the left hand side, where by construction of k we have:

$$\begin{aligned} n &< (k+2)^2 - 1 \\ &= k^2 + 4k + 3 \\ &= (2k^2 - 4k) - (k^2 - 8k - 3) \\ &= 2k(k-2) - [(k-9)(k+1) + 6] \\ &< 2k(k-2) \end{aligned}$$

This confirms that $n \leq 2k(k-2)$ which is the first claimed inequality. The next two claimed inequalities:

$$2k(k-2) < 2k^2 + 1 < 2k(k+2)$$

reduce to $-4k < 1 < 4k$ which are trivially true for $k \geq 9$. For the final inequality, the lower bound for n in the construction of k ensures that:

$$n \geq (k+1)^2 - 1 = k^2 + 2k = k(k+2)$$

Doubling, we have $2k(k+2) \leq 2n$ which was the last part of Step 2 to prove.

Remark: the observation that $(k-9)(k+1) + 6 > 0$ in the proof of the left hand inequality was the same argument we used in the longer proof to demonstrate overlaps between proven intervals for n.