Geometry 9th March 2019

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Some notation which we'll use

In a Δ ABC,

we denote the medians by $[AA']$, $[BB']$ and $[CC']$,

" " altitudes " $[AA_H]$, $[BB_H]$ " $[CC_H]$,

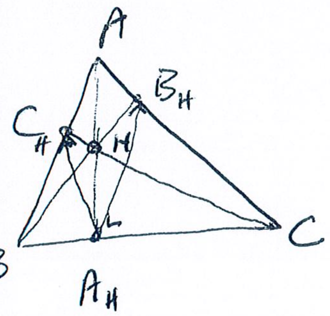
" " circumcentre by G

" orthocentre by H

" circumcentre by O

" incentre by I.

Problem: Using the points marked in the diagram, how many cyclic quadrilaterals can you list. Say why in each case. Name the diameter of the circumcircle of the c.q. if possible.



Answer: (a) $C_H B_H BC$ is a c.q. since $\angle BC_HC = 90^\circ = \angle BB_HC$.

Similarly $B_H A_H BA$ and $A_H C_H AC$ are c.qs.

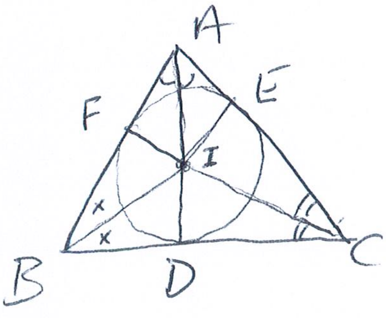
The circumcircles of these three have diameters $[BC]$, $[BA]$, $[AC]$ respectively, since the angle in a semicircle is 90°

(b) $C_H H A_H B$ is a cyclic quadrilateral since $\angle HA_HB = 90^\circ = \angle BC_HH$, with circumcircle diameter $[BH]$

Similarly, $B_H H A_H C$ is " " " $[HC]$

and $B_H H C_H A$ " " " " $[AH]$

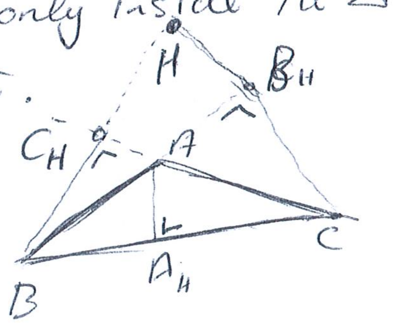
The incircle of a triangle is the circle that has each of the three side of the $\triangle$ as tangents.



The incentre of $\triangle ABC$ is the point where the bisectors of its angles intersect.

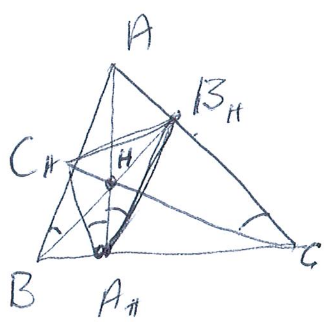
Proof: Every point on the bisector of $\angle ABC$ is equidistant from the arms of the angle i.e. from BA and BC. So $|IF| = |ID|$
Similarly, $|IE| = |ID|$ since IC bisects $\angle ACB$.
So the circle with radius $|IE| = |ID| = |IF|$ touches each of the three sides of $\triangle ABC$.

Note that the orthocentre of a $\triangle$ is only inside the $\triangle$ when all angles of the $\triangle$ are acute.



We show that the orthocentre of the acute-angled $\triangle ABC$ is the incentre of its orthic $\triangle A_H B_H C_H$.

Proof: Let 1 denote $\angle C_H B H$
" 2 " $\angle C_H A_H H$
" 3 " $\angle B_H A_H H$
" 4 " $\angle B_H C H$



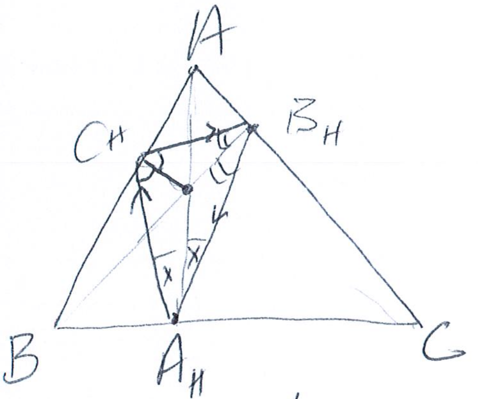
$1 = 2$ since $C_H H A_H B$ is cyclic (Problem page 1) (b)
$3 = 4$ " $H B_H C A_H$ " " " (b)
$1 = 3$ " $C_H B_H C B$ " " " (a)
$\Rightarrow 2 = 3 \Rightarrow A_H H$ bisects $\angle C_H A_H B_H$.
Similarly $B_H H$ " $\angle C_H B_B A_H$
and $C_H H$ " $\angle A_H C_H B_H$.
Thus, H is the incentre of $\triangle A_H B_H C_H$

Some observations about $\Delta A_H B_H C_H$, the <u>orthic</u> Δ of the acute-angled ΔABC.

1) For a ray of light hitting a plane (flat) mirror, the angle of incidence equals the angle of reflection.

Now, imagine a billiard table whose sides form the ΔABC.

A billiard ball travelling towards the point C_H in the direction $A_H C_H$ will bounce off side AB in the direction $C_H B_H$, hit the point B_H, bounce and hit A_H, then C_H again and thus follow a closed billiard path.



2) Can you find <u>another</u> three points on the sides of the ΔABC that form a closed billiard path? Nobody knows whether or not another such closed path exists for general acute-angled triangles. It is an open question.

3) Maryam Mirzakhani (1977–2017) who got a Fields Medal in 2014 (the only woman ever to get one) (and also got a gold medal at the IMO in 1994 and 1995 when she was on the Iranian team) did work in this area.

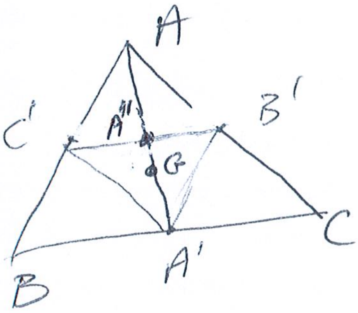
4) $\Delta A_H B_H C_H$ is also known as the Fagnano Δ of ΔABC. It is the Δ, whose vertices are on the sides of ΔABC, with the shortest perimeter. (Fagnano, 17)

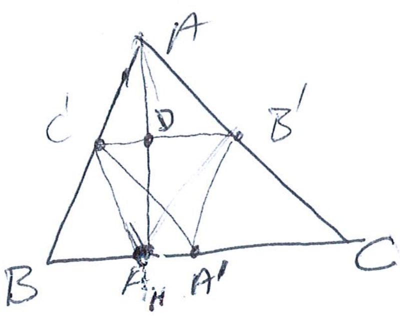
The $\triangle A'B'C'$ is called the <u>medial</u> $\triangle$ of $\triangle ABC$.

Exercises: 1) Show that $\triangle ABC$ and $\triangle A'B'C'$ have the same centroid.

2) Show that $|AG| = 2|GA'|$.

1) Hint: show $AB'A'C'$ is a parallelogram. Then A'', the point where the diagonals meet is the midpoint of $[C'B']$.





We will show that A_H, B_H and C_H are on the circumcircle of the medial $\triangle A'B'C'$.

Proof: If A_H is on the circumcircle of $\triangle A'B'C'$ then $C'B'A'A_H$ is a cyclic quadrilateral so this is what we'll show.

First, $AB'A'C'$ is a parallelogram so

$$\boxed{\angle C'AB' = \angle C'A'B'. \ *}$$

, $C'B' \parallel BC$ and, C' and B' are the midpts of $[AB]$ and $[AC]$ resp,

so $\triangle ADC'$ and $\triangle A_HDC'$ are congruent, [$|AD| = |DA_H|$, $\angle ADC' = \angle A_HDC'$, $|C'D| = |C'D|$]

and so $\boxed{\angle C'AD = \angle C'A_HD \ (1)}$

Similarly, $\triangle AB'D$ and $\triangle A_HB'D$ are congruent so $\boxed{\angle B'AD = \angle B'A_HD \ (2)}$

Adding (1) and (2), we see that $\boxed{\angle B'AC' = \angle C'A_HB' \ **}$

From $*$ and $**$, $\underline{\angle C'A_HB' = \angle C'A'B'}$

Thus, $C'B'A'A_H$ is a cyclic quadrilateral.

Similarly, it follows that B_H and C_H are also on the circumcircle of $\triangle A'B'C'$.