SOLVING COMBINATORIC PROBLEMS

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26 January 2019

What is Combinatorics?

Combinatorics is involved with:

- The enumeration (counting) of specified structures, sometimes referred to as arrangements or configurations, associated with finite systems,
- The existence of such structures that satisfy certain given criteria,
- The construction of these structures, perhaps in many ways, and
- Optimisation, finding the 'best' structure or solution among several possibilities, be it the largest, smallest or satisfying some other optimality criterion.

Thinking about Possibilities

Question: (BMO 17/01/2001 Q3) A *tetromino* is a figure made up of four unit squares connected by common edges.

(1) If we do not distinguish between possible rotations of a tetromino within its plane, prove there are seven distinct tetrominoes.

(2) Prove or disprove the statement: It is possible to pack all seven distinct tetrominoes into a 4×7 rectangle without overlapping.

Hint: Consider special cases according to the longest straight line of squares.

Hint: Consider colouring the rectangle in black and white as a chessboard

Permutations and Combinations:

A **permutation** of the the numbers $1, 2, \ldots n$ is a list of the same numbers, without repeats, in some order.

The number of permutations is:

$$n! = 1 \times 2 \times \ldots \times n$$

Proof: To fill the first place in the list, we can choose from any of the n numbers.

To fill the second place, we can choose any of the remaining $n-1$ numbers, giving $n(n-1)$ choices for the first two places.

To fill the third place, we can choose any of the $n-2$ numbers not yet allocated, giving $n(n-1)(n-2)$ choices for the first three places.

An so the pattern continues until there is only one number left to fill the last place. The total number of choices we have made is $n(n-1)(n-2)\ldots 1$ which is $n!$.

By convention, $0! = 1$. We do not define $n!$ for $n < 0$.

Factors of $n!$

Problem: The number 30! ends in how many zeroes? (in decimal notation)

Solution: Suppose 30! ends in k zeros. Then $10^k \mid 30!$ but $10^{k+1} \nmid 30!$.

Equivalently, $2^k \mid 30!$ and $5^k \mid 30!$ but either $2^{k+1} \nmid 30!$ or $5^{k+1} \nmid 30!$.
So let us count what power of 5 divides 30!. The product contains 6 multiples of 5 (ie 5, 10, 15, 20, 25, 30) but 25 is also multiple of 5^2. Thus, counting the powers in the products, $5^7 \mid 30!$ but $5^8 \nmid 30!$.
It is easy to see that $2^{15} \mid 30!$ so the power of 2 is not a constraint in determining what power of 10 divides 30!.
And therefore, based on the powers of 5, we deduce $10^7 \mid 30!$ and $10^8 \nmid 30!$. It follows that $k = 7$.

Permutations with Duplicates

Problem: How many permutations are there of the letters in

'MARY ROBINSON'?

Solution: If all the letters were distinct, there would be 12! = 479,001,600 permutations.

We note 'MARY ROBINSON' has 2 letter R's, 2 letter O's and 2 letter N's. If for each pair of duplicate letters we colour one red and one blue, then we still have 12! permutations.

If we do not distinguish red and blue letters, then the 12! counts each permutation $8 = 2! \times 2! \times 2!$ times.

Therefore, the number of permutations of 'MARY ROBINSON' is:

$$\frac{12!}{2! \times 2! \times 2!} = 59,875,200$$

Remark: BRAINY MORONS.

Problem: Roisin, Seamus and Terry book seats on Ryanair flight on which seating is allocated randomly. Ryanair uses Boeing 737-800 aircraft. On the left of the plane there are 32 rows of three seats each, while on the right there are 31 rows of three seats each. What is the probability that all three friends are sat together in the same row, on the same side?

News Irish News

You have 1-in-6 MILLION chance of being sat together on a Ryanair flight

By George Morahan - 14/03/2018

MORE NEWS

Solution: We calculate probabilities as proportions of permutations satisfying a given condition.

Roisin can be allocated any of the 189 available seats. There are two other seats in the same row. Let's paint one red and one green. All the others are blue.

There is a 1-in-188 chance that Seamus us allocated the red seat. Following this, there is a 1-in-187 chance that Terry is allocated the green seat, that is 1-in-35,156 that Seamus gets the red seat and Terry the green seat.

But the friends would also be together if Seamus gets the green seat and Terry gets the red seat.

So the total probability that Roisin, Seamus and Terry sit together is:

$$\frac{1}{35,156} + \frac{1}{35,156} = \frac{1}{17,578}$$

Problem: (BMO 12/01/2000, Q5). The seven dwarfs decide to form four teams to compete in the Millenium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy and Grumpy, and one of Bashful and Sneezy. In how many ways can the four teams be made up? (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams).

Hint: Label the teams red, yellow, green, blue. First count how many ways we can make up these four teams when some of the teams are empty.

The original problem required non-empty teams so work out how many of those combinations have one or more empty teams.

Then divide the answer by 24 because we're told the order of teams does not matter.

Number of Combinations

Suppose we have a set of n distinct objects, with $n \geq 2$.

How many ways are there of choosing a subset of size r, for $1 \leq r \leq n-1$?

We can follow the same logic as constructing the permutations:

- Place the elements of the subset in some order.
- The first element of the subset can be chosen in n ways.
- The second element of the subset can be chosen $n-1$ ways, so there are $n(n-1)$ choices for the first two elements.
- ... and so on, until we have $n-r+1$ choices for the r^{th} element of the subset. The number of ways of doing this is:

$$n(n-1)\ldots(n-r+1) = \frac{n(n-1)\ldots 2.1}{(n-r)(n-r-1)\ldots 2.1} = \frac{n!}{(n-r)!}$$

- **But** there were $r!$ ways of choosing the order of the subset elements, so we have counted each subset $r!$ times.
- The number of distinct subsets, not distinguishing different orders, is

$$\binom{n}{r} \equiv \frac{n!}{r!(n-r)!}$$

This is called a *binomial coefficient*.

Pascal's Triangle. We can arrange binomial coefficients in a triangular array:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

Writing the numbers in, and adding a couple more rows, we have the triangle:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

At the boundaries we have:

$$\binom{n}{0} = \binom{n}{n} = 1$$

In the interior of Pascal's triangle, each coefficient is the sum of the two elements above it:

$$\begin{aligned}\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\ &= \frac{r(n-1)!}{r!(n-r)!} + \frac{(n-r)(n-1)!}{r!(n-r)!} \\ &= \frac{(r+n-r)(n-1)!}{r!(n-r)!} \\ &= \binom{n}{r}\end{aligned}$$

The recurrence relation also has a combinatorial interpretation. Suppose a set of n balls has $n-1$ green balls and one red ball.

The subsets of size r consist of:

- Subsets containing the red ball. This involves choosing $r-1$ more balls out of the green balls, which can be done in $\binom{n-1}{r-1}$ ways.
- Subsets not containing the red ball, of which there are $\binom{n-1}{r}$.

Problem: (BMO 01/12/2006 Q3) The number 916238457 is an example of a nine-digit number which contains each of the digits 1 to 9 exactly once. It also has the property that the digits 1 to 5 occur in their natural order, while the digits 1 to 6 do not. How many such numbers are there?

Solution 1: Let us first consider the problem ignoring the constraint on the number 6, so we count the numbers for which 1,2,3,4,5 are in their natural order.

We have 9 digits to place in 9 spaces. Let us place them in decreasing order:

There are 9 spaces into which we can place the digit 9. There are 8 spaces into which we can place the digit 8. (because one is already full). Theretal number of ways is $9 \times 8 \times 7 \times 6 = 3024$.

Now consider instead a similar problem with the constraint that the digits 1,2,3,4,5,6 be in natural order. By the same logic as the previous case, the number of ways is $9 \times 8 \times 7 = 504$.

Finally, for every case where 1,2,3,4,5 is in natural order (3024 are 7 spaces into which we can place the digit 7. There are 6 spaces into which we can place the digit 6. There then remain 5 spaces into which we can place the digits 1,2,3,4,5. We are told these must be in increasing order to there is only one way of doing this. The to), either 1,2,3,4,5,6 are in natural order (504) or they are not (2520 cases). The answer to the problem is then 2520.

Solution 2: We start with the digits 1,2,3,4,5 and insert the remaining digits into the spaces between them.

To place the digit 6, there are 6 spaces around 1,2,3,4,5; that is, one at each end and four between consecutive digits. However, as 1,2,3,4,5,6 cannot be in natural order, there are only 5 of these spaces into which we can put the number 6.

For placing the number 7, we already placed 6 digits, so there are now 7 spaces from which to choose.

For placing the number 8, we already placed 7 digits so there are now 8 spaces from which to choose.

For placing the number 9, we already placed 8 digits, so there are now 9 spaces from which to choose.

Therefore, multiplying the choice at each stage, the total number of ways is $5 \times 7 \times 8 \ \times 9 = 2520$.

Problem (IMO 1972, Q3): Let m and n be arbitrary non-negative integers. Show that:

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

Solution. Let us write:

$$f(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}$$

If $n = 0$ then

$$f(m,0) = \frac{(2m)!}{(m!)^2}$$

This is a binomial coefficient so is an integer.

For the remaining cases $n \geq 1$, we see that the expression for $f(m,n)$ resembles a binomial coefficient so we try to guess a corresponding recurrence relation. We note that:

$$\begin{aligned} f(m,n-1) &= \frac{(2m)!(2n-2)!}{m!(n-1)!(m+n-1)!} \\ &= (m+n)\frac{(2m)!(2n-2)!}{m!(n-1)!(m+n)!} \\ f(m+1,n-1) &= \frac{(2m+2)!(2n-2)!}{(m+1)!(n-1)!(m+n)!} \\ &= 2(2m+1)\frac{(2m)!(2n-2)!}{(m)!(n-1)!(m+n)!} \\ f(m,n) &= \frac{(2m)!(2n)!}{m!n!(m+n)!} \\ &= 2(2n-1)\frac{(2m)!(2n-2)!}{(m)!(n-1)!(m+n)!} \end{aligned}$$

From this, as $2(2m+1)+2(2n-1)=4(m+n)$, we see that:

$$f(m+1,n-1)+f(m,n)=4f(m,n-1)$$

Inductively on n we may assume $f(m+1,n-1) \in \mathbb{Z}$ and $f(m,n-1) \in \mathbb{Z}$, from which it follows that $f(m,n) \in \mathbb{Z}$, which is what we had to prove.