All-Pay Contests with Constraints

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29 January 2012

Abstract

This paper provides simple closed form formulae for players' expected payoffs in a broad class of all-pay contests where players may have constraints on their actions.

1. Introduction

Often agents make costly irreversible investments in hope of winning a prize. In all-pay contests, the players with the highest scores obtain a prize each but the winners' and the losers' costs of effort are at least partially sunk. Under some generic conditions, Siegel (2009) provides closed form formulae for players' expected payoffs in complete information all-pay contests where players compete for one of several identical prizes. In this paper we generalize Siegel (2009) to include contests where players' actions may be constrained.

All-pay contests are used in many areas of research including rent-seeking, political contests, lobbying, patent races, litigation, job tournaments, sports economics, marketing and advertising competition, competition over college seats in selective universities, etc.[1] In any of these competitive settings, contestants may be faced with constraints. For instance, in the USA a cap on political contributions restricts lobbyists who may be attempting to buy political favors through their political donations. In most of Europe and in Canada politicians and political parties are faced with campaign spending limits. In rent-seeking and R&D contests, participants may have liquidity constraints. In litigation, the plaintiff and the defendant fighting over a favorable court decision have a deadline for collecting evidence. In the labor market, employees aiming to impress for promotion are restricted by a maximum of 24 hours of work in a day. In US professional sports leagues (NBA, NFL, NHL, MLS) teams are constrained with annual salary caps. There are score ceilings in the college admissions process as one cannot exceed 2400 on the SAT.[2]

This paper extends Siegel (2009) to provide generalized expected payoff results in all-pay contests with constraints. Section 2 presents the model. The payoff results in Section 3 are followed by an illustrative application.

2. The Model

In cases where we generalize a named assumption or result in Siegel (2009) - henceforth Siegel - we append "generalized" to the name in order to make the changes clear. In cases where we alter the assumption or result but the change is not a strict generalization of the corresponding item in Siegel we append "modified" to the name.

n players compete for m homogeneous prizes where $0 < m < n$. Each player i simultaneously and independently chooses a score s_i from the set of feasible scores $S_i \subseteq \overline{\mathbb{R}}$. $a_i \in [0, \infty)$ is the initial score of contestant i before he puts forth any effort to improve his score, $a_i = \inf S_i$, and we assume that $a_i \in S_i$. The initial score gives the degree of the headstart advantage of the contestant.

Each of the m players with the highest scores wins one prize. In the case of ties any tie-breaking rule can be used to allocate the prizes among the tied players. Given a profile of scores $s = (s_1, \cdots, s_n)$, player i's payoff is:

$$u_i(s) = P_i(s)\, v_i(s_i) - (1 - P_i(s))\, c_i(s_i) \tag{1}$$

where $P_i(s)$ is player i's probability of winning at profile s. His payoff if he wins is given by $v_i(s_i)$. His payoff if he loses is $-c_i(s_i)$. v_i and c_i are defined $\forall\, s_i \in S_i$. The specification in (1) allows for a wide class of all-pay contests; These include contests with many players, multiple prizes, conditional investments, non-ordered asymmetric cost functions with players who have cost advantages in different ranges of score, and contests with variable rewards where the value of the prize to the player depends on his own score. See Siegel for illustrative examples.

Denote $k_i = \sup S_i$ so that for cases where $k_i < \infty$, k_i is a ceiling on the player's choice of score.[3] This is a departure from Siegel since in Siegel players' sets of feasible scores are not constrained from above. Every $s_i \in [a_i, k_i)$ is assumed to be in S_i but k_i may or may not be in S_i. The introduction of a constraint is without loss of generality as the affinely extended real numbers permit the notation $k_i = \infty$ to represent the absence of a constraint. A player is said to be "*restricted at x*" if $x = k_i$ and one of two conditions are satisfied:

a) $x \in S_i$ and $v_i(x) > 0$

b) $x \notin S_i$ and $\lim_{z \to x^-} v_i(z) > 0$

So, a player is restricted at x if he has a positive value from winning at score at x or approaching x from below, but he is unable to exceed that score due to his constraint.

Constraints are permitted for any, none or all players and at any scores. Hence the paper generalizes Siegel in which players do not have constraints. In these environments the paper provides payoff characterizations under the assumptions stated below. First we need to generalize the four main concepts from Siegel which continue to be key to the analysis.

Definitions:

(*i*) Player i's *generalized reach*, r_i, is the supremum of the feasible scores at which the player's valuation for winning is non-negative, $r_i = \sup \{s_i \in S_i \mid v_i(s_i) \geq 0\}$. Re-index players in any decreasing order of their reach, so that $r_1 \geq \cdots \geq r_m \geq \cdots \geq r_n$.

(*ii*) Player $m + 1$ is the *marginal player*.

(*iii*) The *threshold, T*, of the contest is the reach of the marginal player: $T = r_{m+1}$.

(*iv*) Player i's *generalized power, w_i*, is his valuation of winning at his highest feasible score that is less than or equal to the threshold. Formally, if $a_i \leq T$ let $z = \sup \{s_i \in S_i \mid s_i \leq T\}$ and if $a_i > T$ let $z = a_i$. Player i's generalized power is:

$$w_i = \begin{cases} v_i(z) & \text{if } z \in S_i \\ \lim_{x \to z^-} v_i(x) & \text{if } z \notin S_i \end{cases}$$

The definitions of *reach* and *power* are altered from those in Siegel to permit the possibility that a player or players may be restricted. Note that unlike in the unrestricted case as in Siegel, the power of players with reaches less than or equal to the threshold may be positive. If a player $i > m$ is restricted at his reach, then $w_i > 0$.[4] For instance, consider the contest in Figure 1 with one prize and two players. Player 1 has no constraint. Player 2's valuation of the prize is high but he is financially constrained and cannot achieve a score greater than k_2. The reach of Player 1 is r_1. The reach of Player 2 is k_2. Since $r_1 > k_2$, Player 2 is the marginal player. The threshold of the contest is $T = k_2$. In the contest in Figure 1 the marginal player has a higher power than player 1, $w_2 > w_1 > 0$, which cannot happen in the model without constraints.
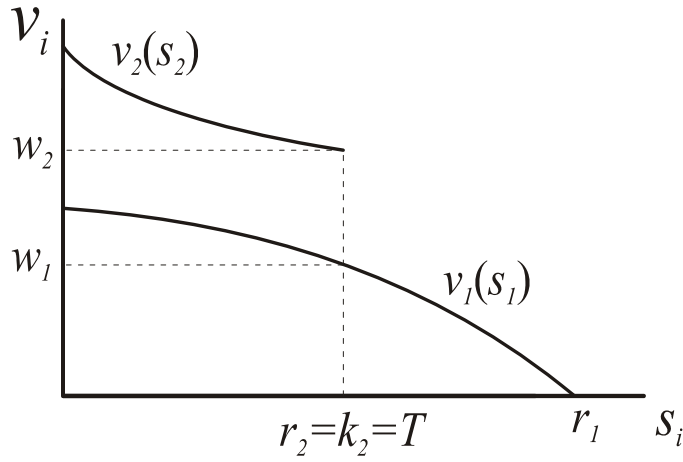

Figure 1

Assumptions:

A1: v_i and $-c_i$ *are continuous and nonincreasing.*.

A2: $c_i(a_i) = 0$, $v_i(a_i) > 0$ *and if* $k_i = \infty$, $\lim_{s_i \to \infty} v_i(s_i) < c_i(a_i) = 0$.

A3: $c_i(s_i) > 0$ *if* $v_i(s_i) = 0$.

Assumptions A1 through A3 are identical to Siegel's. The assumption on v_i in A1 implies that, conditional on winning an increase in the score does not increase the value of the prize by more than the cost of additional effort. A3 and the assumption on c_i in A1 capture the feature of all-pay contests where the winners' and the losers' cost of effort are at least partially sunk. A2 implies that the prize has a strictly positive value for each player and the payoff conditional on winning is

negative with a high enough score. The payoff results are valid for contests that satisfy the following generalized generic conditions:

(i) *Generalized   Power Condition:* The marginal player is the only player with reach at the threshold and players $\{1, \cdots, m\}$ have non-zero power.

(ii) *Generalized   Cost Condition:* If the marginal player is not restricted at the threshold then his valuation from winning is strictly decreasing at the threshold. That is for any $s_{m+1} \in S_{m+1} \bigcap [a_{m+1}, T)$ if $T \in S_{m+1}$ then $v_{m+1}(s_{m+1}) > v_{m+1}(T)$,   and   if $T \notin S_{m+1}$ then $v_{m+1}(s_{m+1}) > \lim_{z \to T^-} v_{m+1}(z)$.

The Generalized Power Condition parallels Siegel's requirement that the marginal player is the only player with power of 0. However with constraints the marginal player may be restricted at the threshold so there may be no player with zero power. It is also possible that a player $i > m+1$ has power zero if $v_i(k_i) = 0$. Therefore with constraints the conditions are not equivalent.

Note that the Generalized Power Condition rules out the cases where the payoff function v_i of any player $i \in \{1, \cdots, m\}$ is zero at the threshold. The Generalized Cost Condition rules out cases where the payoff function of the marginal player, v_{m+1} is flat at zero in the neighborhood below the threshold.

Contests that do not meet the generic conditions can be perturbed slightly to meet them. For instance, if there are two players with the same reach at the threshold, giving one of the players a headstart advantage or the slightest valuation advantage can create a contest that meets the power condition. Likewise, perturbing the marginal player's valuation for winning around the threshold leads to a contest that meets the cost condition.

3. Payoff Characterization

In this section we develop the characterization for the expected payoffs in any equilibrium of any generic contest. This is followed in subsection 3.1 by an example from the literature applying the result and a discussion of the implications of the payoff characterization.

Let $N_w = \{1, \cdots, m\}$ denote the set of players with the m highest reaches. $N_L = \{m + 1, \cdots, n\}$ denotes the set of remaining players, all of whose reaches are less than or equal to the threshold.

Three lemmas are used in the payoff characterization.

Modified Least Lemma: *In any equilibrium of a generic contest, the expected payoff of players in N_L is at least zero and the expected payoff of players in N_w is at least their power.*

Proof: Players in N_L can guarantee a zero payoff by simply choosing a_i. In equilibrium no player would choose a score higher than his reach since such a score is either infeasible or would result in a negative payoff. By the definition of a player's power and the threshold at most m players can have reach greater than T. Since players $i \leq m$ who have $a_i \leq T$ are not restricted at T and are able to exceed the threshold by ε (Assumption A1), they can guarantee at least an expected payoff equal to their power. Players $i \leq m$ who have $a_i > T$ will win with certainty with $s_i = a_i$ by the Power Condition and hence can guarantee a payoff equal to their power.   *Q.E.D.*

To establish the expected payoffs of the players Siegel establishes the Tie Lemma which shows that players either all win with certainty or they all loose with certainty if they choose the score b with strictly positive probability. The Tie Lemma relies on the fact that a player would increase his score slightly to avoid the chance of a tie, if the player's rival has an atom at b and the player can win at b with a positive probability but not with certainty. However if the player is restricted at b this is not possible. Therefore we use an alternative but related method to establish the equilibrium payoffs that does not require an analog of the Tie Lemma.[5]

For each player define G_i as a cumulative probability distribution that assigns probability one to his set of feasible pure strategies S_i. For a strategy profile $G = (G_1, \cdots, G_n)$, $P_i(s_i)$ is player i's probability of winning when he chooses $s_i \in S_i$ and all other players play according to G. Similarly define expected utility $u_i(s_i)$.

Modified Zero Lemma: *In any equilibrium of a generic contest, all players in N_L must have best responses in G with which they win with probability zero or arbitrarily close to zero. These players have expected payoff of zero.*

Proof: Let J denote a set of players including the m players in N_w plus any one other player $j \in N_L$. Let $\tilde{S}$ be the union of the best-response sets of the players in J and let s_{inf} be the infimum of $\tilde{S}$. Consider three cases: (*i*) two or more players in J have an atom at s_{inf}, (*ii*) exactly one player in J has an atom at s_{inf}, and (*iii*) no players in J have an atom at s_{inf}. Examination of these cases help establish the expected payoffs of players in N_L.

Case i. Initially denote $N' \subseteq J$ as the set of all players in J with an atom at s_{inf} where $|N'| > 1$. Every player in $J \setminus N'$ chooses scores greater than s_{inf} with probability 1. Therefore even if every player that is not in J chooses scores strictly below s_{inf} with probability 1 that leaves one too few prizes to be divided between $|N'|$ players, so not all players in N' can win at s_{inf} with certainty.

If there are any players in N' with probability of winning at $s_{\inf}$ equal to 1, remove them from N' so that $P_i(s_{\inf}) < 1$ $\forall$ $i \in N'$. If $|N'|=1$ then that player i loses with certainty with score $s_{\inf}$ and i's expected payoff cannot be positive. From the Modified Least Lemma and the Generalized Power Condition this player cannot be in N_w, so he must be the one player in $J \setminus N_w$, and he must have expected payoff equal to zero. If $|N'| > 1$, then let H be the set $N' \cap N_w$. Since there is only one player in $J \setminus N_w$, $|H| \in \{|N'|-1, |N'|\}$. Probability of winning at $s_{\inf}$ equal to zero, $P_i(s_{\inf}) = 0$, is not possible for any $i \in H$, since i would have $u_i(s_{\inf}) \leq 0$ and he must have a positive payoff by the Modified Least Lemma and the Generalized Power Condition because $H \subset N_w$. If player i loses ties with other players in N' with positive probability, $P_i(s_{\inf}) \in (0,1)$. But this is not possible for any $i \in H$, since i can do better by increasing his score slightly above $s_{\inf}$ to avoid ties by the Generalized Power Condition. Hence at $s_{\inf}$ every player in H must win every tie with other players in N'. This is not possible if $|H| = |N'|$ since there are not enough prizes for all the players in N'. Hence $|H| = |N'|-1$ so $j \in N'$ and j loses all ties with members of N' at $s_{\inf}$. Therefore $P_j(s_{\inf}) = 0$ and $u_i(s_{\inf}) \leq 0$ since $j \in N'$ and $j \in N_L$. By the Modified Least Lemma his expected payoff must be zero.

Cases ii and iii. The corresponding proofs in Siegel apply without modification and establish that in both cases one player $i \in J$ has a best response in which he wins with probability 0 or arbitrarily close to 0 and has an expected payoff of at most 0. By the Modified Least Lemma i must have a payoff of 0, and by the Generalized Power Condition $i \in N_L$ and so $i = j$.

The above applies for each player $j \in N_L$.   Q.E.D.

Generalized Threshold Lemma: *In any equilibrium of a generic contest, the players in N_w have best responses in G that approach or exceed the threshold and, therefore, the players in N_w have an expected payoff of at most their power.*

Proof: The proof is omitted here as the proof of the Threshold Lemma in Siegel applies without modification noting only that with constraints players in $N_L \setminus \{m+1\}$ may or may not have negative powers, however they still have reaches strictly below the threshold.

From these intermediate results we can establish the main result of the paper.

Generalized Theorem 1: *In any equilibrium of a generic contest, the expected payoff of each player in N_w is equal to his power which is greater than zero, and the expected payoff of each player in N_L is zero which is less than his power if he is restricted at his reach.*

Proof: The Modified Least Lemma and the Generalized Threshold Lemma establish that players in N_w have expected payoffs equal to their power which is greater than zero by the Generalized Power Condition. The Modified Zero Lemma establishes that the players in N_L have expected payoffs equal to 0. If a player in N_L is not restricted at his reach, his power is less than or equal to zero. If he is restricted at his reach his power is greater than zero so his expected payoff is less than his power.   *Q.E.D.*

Because players' expected payoffs from the contest depend only on the order of their reaches and on their valuation of winning at the threshold, the striking insight of Siegel continues to hold in contests with constraints; The players' costs of losing and the shapes of v_i and c_i away from the threshold do not affect payoffs. They may have an effect on equilibrium strategies, but not on expected payoffs. Similarly, a change in the constraint of any player other than the marginal player does not affect the expected payoff of any player as long as the change does not alter the identity of the marginal player:

Corollary to Generalized Theorem 1: *In any equilibrium of a generic contest, consider a small change in a player's constraint such that the identity of the marginal player remains the same. A change in the constraint of any player other than the marginal player does not affect the payoff of any player. A change in the marginal player's constraint does not affect the payoff of any player in N_L (including his own). If the marginal player is restricted at k_{m+1} then relaxing his constraint decreases the expected payoff of each player in N_w.*

Proof: All points follow directly from Generalized Theorem 1, the definitions of reach and power. The derivative of players' expected payoffs with respect to k_j is zero for all $j \neq m+1$ since changing k_j does not alter T. For all $i \in N_L$ the derivative of player i's expected payoff with respect to k_{m+1} is zero. If $T = k_{m+1}$ a marginal decrease in k_{m+1} decreases T and a marginal increase in k_{m+1} increases T if player $m+1$ is restricted at T. If $T \neq k_{m+1}$ then marginal changes in k_{m+1} do not alter T. Finally, if $T = k_{m+1}$, then for all $i \in N_w$ the derivative of player i's expected payoff with respect to k_{m+1} is $-\left.\frac{\partial v_i(s_i)}{\partial s_i}\right|_{s_i=T} \leq 0$. If $T \neq k_{m+1}$ then the derivatives of all players' expected payoffs with respect to k_{m+1} are zero.   *Q.E.D.*

One implication of the Corollary is that a player's expected payoff is affected by a change in his own constraint only if the change in his constraint switches him between N_w and N_L. Other changes in his constraint may well affect equilibrium strategies, but they will not affect the player's own payoffs.

3.1. Discussion of the Payoff Characterization.

To illustrate the use of Generalized Theorem 1 consider an example from the literature. To derive players' equilibrium expected payoffs all that is needed is the players' reaches and the power of players in N_w.

Example: Meirowitz (2008) analyzes the sources of incumbency advantage in a first-past-the-post electoral contest where politicians compete in campaign spending. One dollar of campaign spending raises the score of the political candidate by one. The incumbent (I) and the challenger (C) have a common valuation of the prize normalized to 1. The candidates have potentially different marginal utility cost of raising funds, $\beta_i \ \forall i \in \{I, C\}$. In the all-pay contest without spending limits, Meirowitz (2008) considers a positive headstart advantage $\alpha > 0$ for the incumbent due to existing name recognition. When studying the effect of a spending limit $\overline{m}$ the analysis only presents the case where the limit is so restrictive $(\overline{m} < \alpha)$ that even if the challenger where to spend the maximum permissible amount and the incumbent were to spend zero the incumbent would win the contest. Hence the equilibrium is in pure strategies where no candidate engages in campaign spending.

(*i*) **Application of the Generalized Theorem 1 to spending limits with a headstart advantage, 0<α< $\overline{m}$:** Using Generalized Theorem 1 it is straightforward to extend Meirowitz (2008) to analyze less restrictive spending limits.

There are arguments for and against spending limits. Opponents of limits suggest that limits benefit the incumbent by restricting the challenger's ability to catch up with the incumbent who often enjoys a headstart advantage. However proponents of spending limits argue that limits level the playing field in favor of the candidate with lesser resources. Incumbents tend to be more efficient at fund-raising. As a sitting officeholder an incumbent is in a position to dispense political favors and hence has better access to resources, $\beta_I < \beta_C$. Therefore it is often argued that a limit restricts the ability of the incumbent to take full advantage of his fundraising efficiency and hence benefits the challenger.

Generalized Theorem 1 can be applied to show that with any headstart advantage, $\alpha>0$, however small, in any equilibrium a spending limit benefits the incumbent no matter how dramatic the difference in fundraising abilities may be. The "headstart advantage" argument of the opponents of spending limits always trumps the "lesser resources" argument of the proponents of limits.

The first step is to convert Meirowiz' framework into the notation of this paper. The monetary limit on campaign spending is denoted by $\overline{m}$ and is common to both players. However, since the incumbent has a headstart advantage of $a_I = \alpha$ while $a_C = 0$, the constraints on scores are asymmet-

ric: $k_C = \overline{m}$ and $k_I = \alpha + \overline{m}$. The challenger's payoff and cost functions are given by $v_C(s_C) = 1 - \beta_C s_C$ and $c_C(s_C) = \beta_C s_C$ for $s_C \in [0, k_C]$. Since the incumbent starts with a score of α his payoff function is $v_I(s_I) = 1 - \beta_I(s_I - \alpha)$ and $c_I(s_I) = \beta_I(s_I - \alpha)$ for $s_I \in [\alpha, k_I]$. Therefore the reach of the challenger is $r_C = \min\{\overline{m}, 1/\beta_C\}$ and the reach of the incumbent is $r_I = \min\{\alpha + \overline{m}, \alpha + 1/\beta_I\}$.

Without a spending limit, the reach of the incumbent is higher than the reach of the challenger, $1/\beta_C < \alpha + 1/\beta_I$. Hence the challenger is the marginal player. From Generalized Theorem 1 the challenger has zero expected payoff. The threshold is $1/\beta_C$, so the incumbent has an expected payoff equal to his power, $1 - \beta_I(1/\beta_C - \alpha) > 0$.

If the expenditure limit is less than $1/\beta_C$ it becomes binding and $r_C = \overline{m} < \overline{m} + \alpha$. Since r_C is less than the incumbent's reach $r_I = \min\{\alpha + \overline{m}, \alpha + 1/\beta_I\}$ the challenger is still the marginal player and his expected payoff remains zero. However the limit reduces the challenger's reach (the threshold of the game) and hence increases the expected payoff of the incumbent to $1 - \beta_I(\overline{m} - \alpha) > 0$. Hence the imposition of a spending limit always benefits the incumbent as long as the incumbent has a headstart advantage however small that may be, as long as candidates have symmetric campaign spending efficiency.

In addition, the Corollary has implications for *ex ante* investment. Suppose that prior to the above game the two parties had the opportunity to increase their initial score a_i through voter registration drives. Increases in a_I would have a positive benefit for the incumbent but marginal increases in a_C would not benefit the challenger.

(*ii*) **Application of Generalized Theorem 1 to spending limits with multiple candidates and asymmetric campaign spending effectiveness:** Meirowitz (2008) has only two candidates running for election. However in countries such as France, the UK where campaign spending limits are in place with a first-pass-the-post system often more than two political parties compete. The payoff characterization in Generalized Theorem 1 does not require the full derivation of the equilibrium. Hence we can easily add more candidates and compute which political candidate benefits from a spending limit in any equilibrium. Below we employ the theorem to show that a moderate cap may benefit a charismatic third-party candidate, while a very restrictive cap benefits the incumbent. This demonstrates that although the "headstart advantage" argument always dominates the "lesser resources" argument in favor of spending limits if candidates have equal spending efficiency, limits may benefit an opponent if he is more efficient in spending, which is often found empirically.

Add a third-party candidate to the model described in Application (*i*) with the same notation. Suppose that the third-party candidate (candidate L) is charismatic and has leadership skills so that one dollar of campaign spending increases his score by $\eta_L > 1$. In order to restrict attention to interesting cases, assume that his campaign spending efficiency is $\eta_L \in (1, \beta_L(\alpha + 1/\beta_I))$ while

$\eta_I = \eta_C = 1$. As a third party candidate he lacks a large fundraising base so fundraising is weakly more onerous for him than for candidate C. Assume that $\beta_L \in \left[\beta_C, \frac{\eta_L - 1}{\alpha}\right)$. So the third-party candidate's cost of achieving the score s_L is $\frac{s_L}{\eta_L} \beta_L$ and his reach is $r_L = \min\{\eta_L \bar{m}, \eta_L / \beta_L\}$. As before $a_I = \alpha$ is the headstart advantage of the incumbent and all candidates have a common valuation of the prize normalized to 1.

In the absence of a spending limit and under the parameter restrictions above $r_I > r_L$ and $r_I > r_C$. By Generalized Theorem 1, the incumbent has a positive expected payoff while the challenger and the third party candidate receive an expected payoff of zero. The challenger is disadvantaged because the incumbent has a head-start advantage and is a better fundraiser. The third-party candidate has greater effectiveness of campaign spending but this is not enough to overcome the incumbency advantage.

However, with a common monetary cap $\bar{m} < \frac{1}{\beta_L}$, all candidates are restricted at their score caps and the reaches of the candidates are given by $r_I = \alpha + \bar{m}$ and $r_L = \eta_L \bar{m}$ and $r_C = \bar{m}$. If the cap is moderate $\bar{m} \in \left(\frac{\alpha}{\eta_L - 1}, \frac{1}{\beta_L}\right)$, then $r_L > r_I > r_C$. The incumbent becomes the marginal player. The threshold of the contest is $T = \alpha + \bar{m}$. By Generalized Theorem 1, the incumbent and the challenger have expected payoffs of zero and the third-party candidate receives an expected payoff of $1 - \beta_L\left(\frac{\alpha + \bar{m}}{\eta_L}\right) > 0$. With a moderate limit, the effective campaign spending of the third-party candidate overwhelms the head-start advantage of the incumbent. Hence a moderate limit hurts the incumbent compared to no restrictions.

If the cap is very restrictive, $\bar{m} \in \left[0, \frac{\alpha}{\eta_L - 1}\right)$, then the order of candidates' reaches is given by: $r_I > r_L > r_C$. The third party candidate is the marginal player and $\eta_L \bar{m}$ is the threshold. By Generalized Theorem 1, the incumbent has the expected payoff $1 - (\eta_L \bar{m} - \alpha)\beta_I > 0$. The challenger and the third-party candidate have expected payoff zero. The head-start advantage of the incumbent overwhelms the campaign spending effectiveness of the third-party candidate with leadership skills. The cap is too restrictive for the third-party candidate to catch up with the incumbent. Note that the expected payoff of the incumbent in this case is higher than the expected payoff he would have had if there were no campaign spending restrictions.

In both these applications, once the expected payoffs of the players are established it is straightforward to derive equilibrium distributions of the players as well as results on expected spending and probability of winning by employing the approach in Hillman and Riley (1989) and Baye *et. al* (1993) .

4. Conclusion

This paper provides a generalization of Siegel (2009) to include constraints on players' choices. In a broad class of all-pay contests, it derives a simple closed-form solution for expected payoffs in any equilibrium. These include contests with many players, many prizes, non-ordered costs functions, conditional investments, and/or constraints on players' actions. In some applications players' expected payoffs are the main item of interest. For example, one may be concerned about the effect of a policy on the market participants. Or one may be primarily interested in whether agents in a multi-stage game chose to enter a contest. In these cases as long as an equilibrium exists the results can be used directly, bypassing the need for the full derivation of the equilibrium. In applications where the full characterization of the equilibrium is of interest, finding the players' expected payoffs is a crucial first step in the derivation of the equilibrium.

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Notes

[1] Among others see Hillman and Samet (1987), Hillman and Riley (1989), Ellingsen (1991), Baye *et. al* (1993), and Konrad (2002) for contests in rent seeking; Che and Gale (1998 and 2006), Kaplan and Wettstein (2006), Meirowitz (2008), and Pastine and Pastine (2010b) for political contests; Leininger (1991) for R&D races; Bond (2009) for litigation contests; Clark and Riis (1998) for job tournaments; Pastine and Pastine (2011) for advertising competition; Fu (2006) and Pastine and Pastine (2012a) for affirmative action in college admissions. See Konrad (2009) for an extensive survey on contests.

[2] For all-pay contests with constraints see Che and Gale (1998), Meirowitz (2008), Pai and Vohra (2009), Megidish and Sela (2010), Pastine and Pastine (2010a and 2012b), Szech (2011) and Leininger (1991) in complete-information frameworks. See Che and Gale (1996), Laffont and Robert (1996), Gavious, Moldovanu and Sela (2002), Sahuguet (2006), Kirkegaard (2008), Sahuguet (2006) and Pai and Vohra (2009) in frameworks with incomplete information. See Rapoport and Amaldoss (2000) for an experimental analysis of all-pay auctions with bid caps and the comment in Dechenaux *et. al* (2006).

[3] Conceptually there are two possible types of constraints: constraints on effort and constraints on scores. Both types of constraints can be captured by this specification since effort translates into scores. With constraints directly on scores, an initial score higher than the maximum possible score is nonsensical. So $k_i \geq a_i$ can be assumed without loss of generality.

[4] In Siegel the power is defined as a player's valuation from winning at the threshold. However with constraints this would require that $v_i(T)$ and $c_i(T)$ be defined for cases where $T \notin S_i$ which would raise non-trivial conceptual issues. What is the cost of a loan that nobody is willing to extend? Moreover it would not alter the fact that a competitor who has high valuation but is constrained at a score below the threshold would have positive power.

[5] Employing the Tie Lemma, Siegel presents two results (Corollaries 2 and 3) which hold for non-generic contests without constraints; Corollary 2 shows that every continuous non-generic contest has at least one equilibrium in which the expected payoffs match the continuous-generic-game expected payoffs given by Siegel's Theorem 1. Corollary 3 shows that in any contest without constraints where all players are identical, all players have an expected payoff of zero. However these corollaries do not extend to non-generic contests with constraints. For instance in the lobbying contest of Che and Gale (1998) for a sufficiently restrictive common constraint, k, in any equilibrium both players choose the score equal to the constraint with probability 1. This contest is non-generic because for low k, players have the same reach, violating the Power Condition. Players' expected payoffs are strictly greater than zero and less than their power violating a conjectured extension of Siegel's Corollary 2. The same logic carries over to identical players facing a common constraint

yielding positive expected payoffs for both players, a violation of a conjectured extension of Siegel's Corollary 3. These conjectured extensions fail because with constraints in equilibrium players can put probability mass points at scores where they do not win or lose with certainty. This is precisely the reason that Siegel's Tie Lemma does not extend to contests with constraints.

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